# LECTURE NOTES ON ANALYTIC GEOMETRY AND CALCULUS

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## ABSTRACT

Geometry is a branch of mathematcs concerned with questions of shapes, size, relative position of figures and properties of space. Some general concepts that are fundamental to Geometry are points, lines, planes, surfaces, angles, curves as well as topology or metric.

Analytic geometry also referred to as Coordinate geometry or Cartesian geometry is the study of geometry using coordinate system. Analytic geometry is a branch of algebra that is used to model geometric objects such as points, straight lines and circles being the best basic of these.

In plane analytic geometry, points are defined as ordered pairs of numbers, say $(x, y)$, while the straight lines are in turn defined as the set of points that satisfy linear equation.

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# Chapter 1

# COORDINATE GEOMETRY

Points on a line can be identified with real numbers to form the `Coordinate line`. Similarly, points on a plane can be identified with ordered pairs of numbers to form the `Cartesian plane`. For instance, any point $P$ in the coordinate plane can be located by a unique ordered pair of numbers $(a, b)$. The first number $a$ is referred to as the $x$-coordinate of $P$ and the second number $b$ is also referred to as the $y$-coordinate of $P$.

## 1.1 Formula for the Distance $d(A, B)$ between two points $A(x_1, y_1)$ and $B(x_2, y_2)$

Let us consider the right-angled triangle $ABC$ as shown in Figure 1.1. By using the `Pythagoras Theorem`[1], we obtain

$$d(A, B)^2 = |AC|^2 + |BC|^2$$
$$d(A, B)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Therefore, the equation for the distance between two points is

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \qquad (1.1)$$

**Example 1.1** *Which of the points $P(-1, -2)$ or $Q(8, 9)$ is closer to the point $A(3, 5)$?*

[1]The Pythagoras theorem : $z^2 = x^2 + y^2$ where $z$ is the hypotenuse of a right-angled triangle.

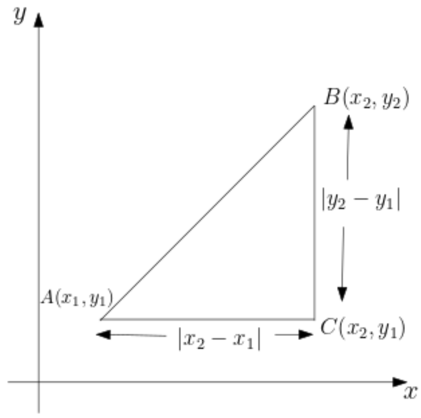



Figure 1.1: Illustration for the derivation of the distance formula.

**Solution**

*By using the distance formula* (1.1), *we determine the distances* $PA$ *and* $QA$ *such that*

$$\begin{aligned} d(P, A) &= \sqrt{(3 - (-1))^2 + (5 - (-2))^2} \\ &= \sqrt{4^2 + 7^2} \\ &= \sqrt{65}, \end{aligned}$$

*and*

$$\begin{aligned} d(Q, A) &= \sqrt{(3 - 8)^2 + (5 - 9)^2} \\ &= \sqrt{(-5)^2 + (-4)^2} \\ &= \sqrt{41}. \end{aligned}$$

*Since* $d(Q, A) < d(P, A)$ *we conclude that* $Q$ *is closer to* $A$.

## 1.2 Midpoint of a line segment from $A(x_1, y_1)$ to $B(x_2, y_2)$

*Let us consider the Figure 1.2, if* $M$ *is the midpoint of the line segment* $AB$, *then we have equal distances along the* $x-$*coordinate as follows*

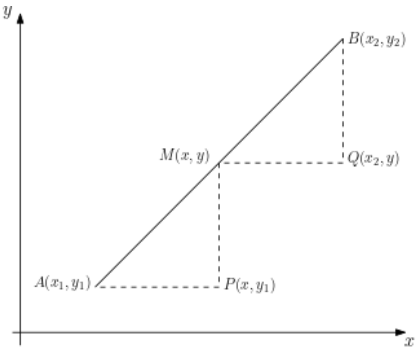



Figure 1.2: Illustration for the derivation of the midpoint formula.

$$d(A, P) = d(M, Q).$$

*Therefore, by change of subject, we have*

$$\begin{aligned} x - x_1 &= x_2 - x \\ 2x &= x_2 + x_1. \end{aligned}$$

*Finally, we obtain*

$$x = \frac{x_2 + x_1}{2}.$$

*Also, we have equal distances along the y−coordinate as follows*

$$d(P, M) = d(Q, M).$$

*Therefore, by change of subject, we have*

$$\begin{aligned} y - y_1 &= y_2 - y \\ 2y &= y_2 + y_1. \end{aligned}$$

*Finally, we obtain*

$$y = \frac{y_2 + y_1}{2}.$$

*The mid-point formula for a line segment is thus given by*

$$M\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right). \tag{1.2}$$

***Example 1.2*** *Show that the equilateral with vertices $P(1, 2), Q(4, 4), R(5, 9)$ and $S(2, 7)$ is a parallelogram by proving that its two diagonals bisect each other.*

**Solution**

*If the two diagonals have the same midpoint, then they must bisect each other as shown in Figure 1.2. Therefore, using the*

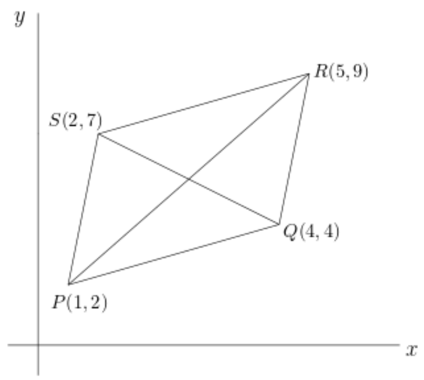



Figure 1.3: Illustration of the parallelogram $PQRS$.

*midpoint formula* (1.2)*, we find the midpoint of the diagonal* $PR$ *as*

$$\left(\frac{1+5}{2}, \frac{2+9}{2}\right) = \left(3, \frac{11}{2}\right),$$

*and the midpoint of the diagonal* $QS$ *is given as*

$$\left(\frac{2+4}{2}, \frac{7+4}{2}\right) = \left(3, \frac{11}{2}\right).$$

*Since the coordinates of* $PR = QS$*, it implies the two diagonals bisect each other.*

## 1.3 Gradient of a Line

*We consider a triangle* $ABC$ *with coordinates* $A(x_1, y_1), B(x_2, y_2)$ *and* $C(x_3, y_3)$ *as shown in Figure 1.4.*

*The* ***Gradient*** *of the line* $AB$ *is given by* $AB = \dfrac{y_2 - y_1}{x_2 - x_1}$. *Also, since we have a right-angled triangle, by using the* ***trigonometic identities****, the gradient of* $AB$ *can be written as* $AB = \tan\theta$.

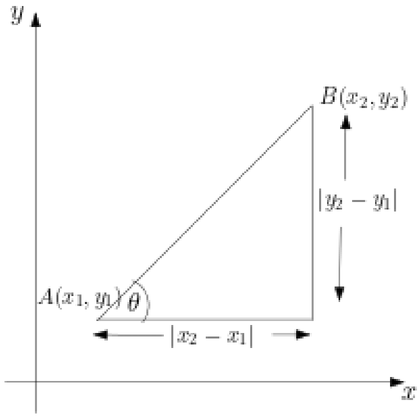



Figure 1.4: Illustration of the Gradient formula

**Relationship Between the Gradients of Parallel Lines**

*Let two lines $L_1$ and $L_2$ have gradients $m_1$ and $m_2$, respectively. If $L_1$ and $L_2$ are parallel then $m_1 = m_2$, i.e. Parallel lines have equal gradients.*

**Relationship Between the Gradients of Perpendicular Lines**

*If the lines $L_1$ and $L_2$ with gradients $m_1$ and $m_2$ are perpendicular, then $m_1 * m_2 = -1$ thus $m_2 = -\frac{1}{m_1}$.*

## 1.4 Equation of a Line

*Consider a line through the point $A(x_1, y_1)$ with gradient $m$. Let $B(x, y)$ be any other point on the line as shown in Figure 1.5.*

$$m = \frac{y - y_1}{x - x_1}$$
$$y - y_1 = m(x - x_1). \quad (1.3)$$

***Example 1.3*** *Find the equation of the line through A(1,3) with gradient 5.*

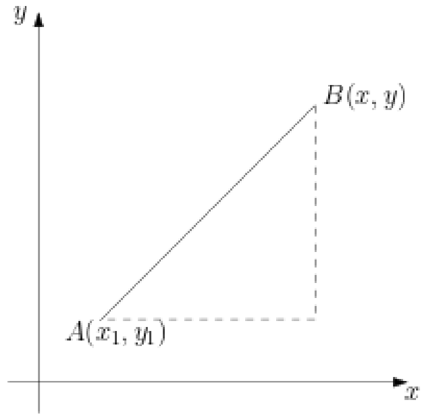



Figure 1.5: Illustration of the equation of a line.

**Solution**

*Let* $x_1 = 1, y_1 = 3$, *and the gradient* $m = 5$, *then by using* (1.3), *the equation of the line is given by*

$$\begin{aligned} y - 3 &= 5(x - 1) \\ y &= 5x - 5 + 3 = 5x - 2. \end{aligned}$$

*Sometimes the equation of the line is expressed in the form* $y = mx + c$, *in such situations the gradient* $m$ *of the line is the coefficient of* $x$.

## 1.5 Angles Between Two Lines

*Consider the lines* $L_1$ *and* $L_2$ *with gradients* $m_1$ *and* $m_2$, *respectively. Suppose that* $m_1 > m_2 > 0$.

*Observe that there are two distinct angles between* $L_1$ *and* $L_2$. *These angles are* $\theta$ *and* $(180 - \theta)$.
*Also, from the figure above,*

$$\lambda = 180 - \alpha \qquad \textit{and} \qquad \lambda = 180 - \beta - \theta. \tag{1.4}$$

*From (1) and (2), we obtain,*

$$180 - \alpha = 180 - \beta - \theta$$

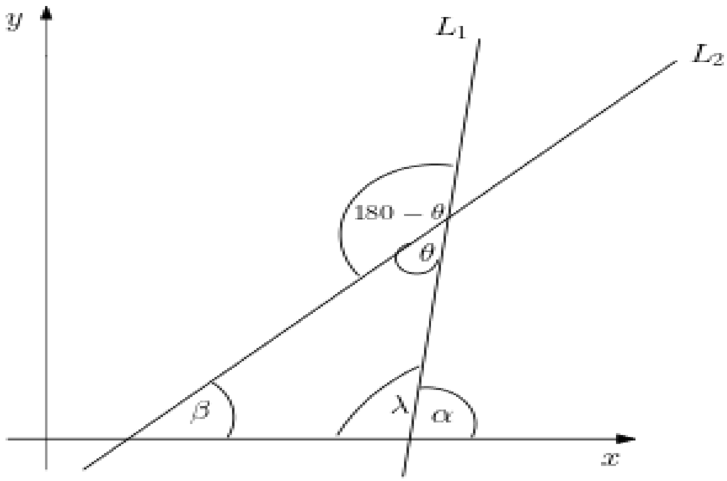



Figure 1.6: Angle between two lines formula.

$$\theta = \alpha - \beta.$$

*Thus,*

$$\begin{aligned}\tan\theta &= \tan(\alpha - \beta)\\ &= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.\end{aligned}$$

*Therefore,*

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} \tag{1.5}$$

*Also,*

$$\begin{aligned}\tan(180 - \theta) = \frac{\tan 180 - \tan\theta}{1 + \tan 180 \tan\theta} &= -tan\theta\\ &= -\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right).\end{aligned}$$

***Example 1.4*** *Find the acute angle $\theta$ between the lines $y = 2x + 1$ and $y = 3x + 2$.*

**Solution**

*Let* $m_1 = 3$ *and* $m_2 = 2$*. Then,*

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{3-2}{1+3(2)} = \frac{1}{7}$$
$$\theta = \tan^{-1}(0.1429) = 8.1325^o.$$

***Example 1.5*** *A point* $A(1, 2)$ *is one vertex of a parallelogram* $ABCD$*. Side* $AB$ *lies on the line* $2y = x + 3$ *and the diagonal* $BD$ *lies on the line* $3y + x = 17$*. If* $|AD| = |BD|$*. Find the coordinate of* $C$*. Find also the equation of the perpendicular bisector of* $AB$*.*

**Solution**

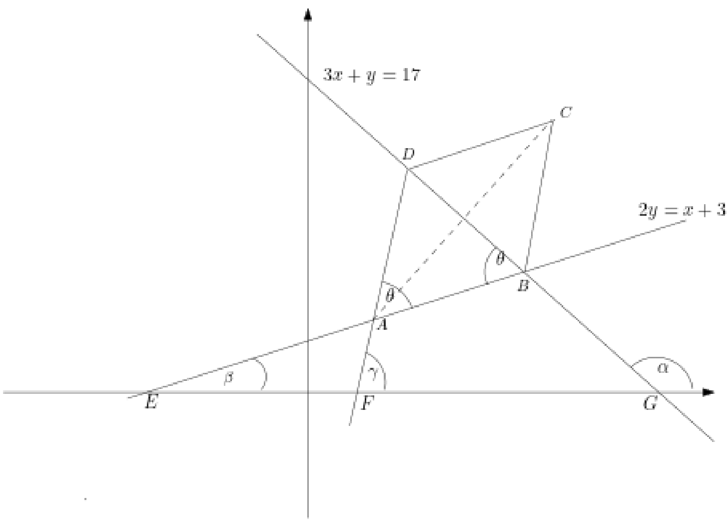



Figure 1.7: Illustration of Example 1.5.

*Let* $\angle DBA = \theta$ *as illustrated in Figure 1.7. We first find the coordinates of* $B$ *and* $D$ *as follows: From* $\triangle EAF$*,*

$$\gamma = \theta + \beta$$
$$\tan\gamma = tan(\theta + \beta)$$
$$= \left(\frac{\tan\theta + \tan\beta}{1 - \tan\theta\tan\beta}\right)$$

$$but \quad \tan\beta = \frac{1}{2}$$

*Also, from* $\triangle EBG$

$$\begin{aligned}\beta + 180 - \theta + 180 - \alpha &= 180\\ 180 - \theta &= \alpha - \beta\\ \tan(180 - \theta) &= \tan(\alpha - \beta)\\ \therefore -\tan\theta &= \frac{-1/3 - 1/2}{1 + (-1/3)(1/2)} = \frac{-5/6}{5/6} = -1.\end{aligned}$$

*Thus,*

$$\tan\gamma = \frac{1 + 1/2}{1 - (1)(1/2)} = \frac{3/2}{1/2} = 3.$$

*The gradient of* $\overline{AD}$ *is* 3. *Hence, the equation of* $\overline{AD}$ *is given by*

$$\begin{aligned}y - 2 &= 3(x - 1)\\ y &= 3x - 1. \end{aligned} \tag{1.6}$$

*By simultaneously solving*

$$y = 3x - 1 \qquad and \qquad x + 3y = 17,$$

*we obtain* $y = 5$ *and* $x = 2$ *yielding the point* $D(2, 5)$. *Also, by solving*

$$2y = x + 3 \qquad and \qquad 3y = -x + 17,$$

*simultaneously, we obtain* $y = 4$ *and* $x = 5$ *yielding the point* $B(5, 4)$. *Let* $M$ *be the midpoint of* $\overline{BD}$ *and also the midpoint of* $\overline{AD}$. *Then* $M$ *is given by* $M\left(\frac{7}{2}, \frac{9}{2}\right)$. *Next, we find the coordinates of* $C$ *as follows. Let* $C$ *be* $(p, q)$, *then,*

$$\frac{1 + p}{2} = \frac{7}{2} \implies p = 6, \qquad and \qquad \frac{q + 2}{2} = \frac{9}{2} \implies q = 7.$$

*Hence, the coordinates of $C$ is* $(6, 7)$.

*Finally, to find the equation of the perpendicular bisector of $AB$, we have to find the midpoint of $AB$, given by*

$$\textit{Midpoint of } AB = \left(\frac{5+1}{2}, \frac{4+2}{2}\right) = (3, 3).$$

*The gradient of the perpendicular line to $AB$ is* $m = \dfrac{-1}{1/2} = -2$. *Thus, the equation of the perpendicular bisector is*

$$\begin{aligned} y - 3 &= -2(x - 3) \\ y &= -2x + 9. \end{aligned}$$

## 1.6 Division of a line segment in a given ratio

*We will discuss about the internal and external division of line segment. To find the coordinates of the point dividing the line segment joining two given points in a given ratio.*

### 1.6.1 Internal Division

*Consider the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$. Let $R(x, y)$ divides $AB$ internally in the ratio $n : m$, as shown in Figure 1.8.*

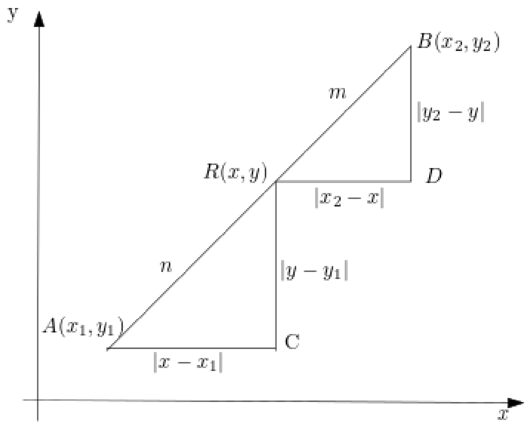



Figure 1.8: Illustration of internal division of a line segment.

*Since triangle $ARC$ and $RBD$ are similar, we proceed as follows*

$$\begin{aligned}
\frac{|AC|}{|RD|} &= \frac{n}{m} \\
\frac{x - x_1}{x_2 - x} &= \frac{n}{m} \\
m(x - x_1) &= n(x_2 - x) \\
mx - mx_1 &= nx_2 - nx \\
x(m + n) &= nx_2 + mx_2 \\
x &= \frac{nx_2 + mx_1}{n + m}.
\end{aligned}$$

*Also,*

$$\begin{aligned}
\frac{|RC|}{|BD|} &= \frac{n}{m} \\
\frac{y - y_1}{y_2 - y} &= \frac{n}{m} \\
m(y - y_1) &= n(y_2 - y) \\
my - my_1 &= ny_2 - ny \\
y(m + n) &= ny_2 + my_1 \\
y &= \frac{ny_2 + my_1}{m + n}.
\end{aligned}$$

*Thus, the coordinates of the point $R$ is*

$$R\left(\frac{mx_1 + n_2}{m + n}, \frac{my_1 + ny_2}{m + n}\right).$$

### 1.6.2 External Division

*Suppose the point $R$ divides the line $AB$ externally such that $|AR| : |RB| = -n : m$ as illustrated in Figure 1.9.*

*Since triangle $ARC$ and $RBD$ are similar, we have*

$$\frac{|RC|}{|RD|} = \frac{n}{m}$$

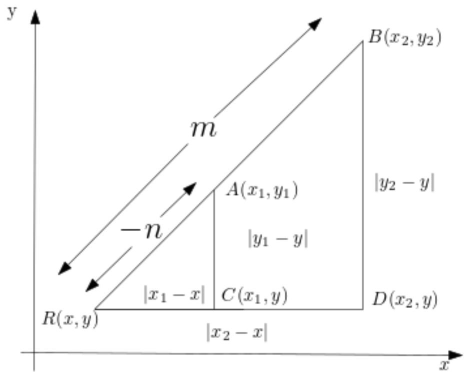



Figure 1.9: Illustration of external division of a line segment.

$$\begin{aligned}
\frac{x_1 - x}{x_2 - x} &= \frac{n}{m} \\
n(x_2 - x) &= m(x_1 - x) \\
mx - nx &= mx_1 - nx_2 \\
x(m - n) &= mx_1 - nx_2 \\
\therefore \quad x &= \frac{mx_1 - nx_2}{m - n}.
\end{aligned}$$

*Also,*

$$\begin{aligned}
\frac{|AC|}{|BD|} &= \frac{n}{m} \\
\frac{y_1 - y}{y_2 - y} &= \frac{n}{m} \\
m(y_1 - y) &= n(y_2 - y) \\
my_1 - my &= ny_2 - ny \\
my - ny &= my_1 - ny_2 \\
y(m - n) &= my_1 - ny_2 \\
\therefore \quad y &= \frac{my_1 - ny_2}{m - n}.
\end{aligned}$$

*Thus, the coordinates of the point $R$ is*

$$R\left(\frac{mx_1 - nx_2}{m - n}, \frac{my_1 - ny_2}{m - n}\right).$$

*Similarly, if $R$ divides the line $AB$ externally such that $|AR| : |RB| = n : -m$ then,*

$$R\left(\frac{-mx_1 + nx_2}{-m+n}, \frac{-my_1 + ny_2}{-m+n}\right).$$

***Example 1.6*** *Find the points dividing the join of $A(1, 2)$ and $B(3, 1)$.*

*a. Internally in the ratio $1 : 2$*

*b. Externally in the ratio $-1 : 2$*

**Solution**

*a. Let $(a, b)$ be the point of internal division. Then*

$$\begin{aligned}(a, b) &= \left(\frac{mx_1 + nx_2}{m+n}, \frac{my_1 + ny_2}{m+n}\right) \quad \textit{where} \quad n : m = 1 : 2 \\ &= \left(\frac{2(1) + 1(3)}{1+2}, \frac{2(2) + 1(1)}{1+2}\right) = \left(\frac{5}{3}, \frac{5}{3}\right).\end{aligned}$$

*Therefore, the required point is* $\left(\frac{5}{3}, \frac{5}{3}\right)$.

*b. Let $(p, q)$ be the point of external division. Then*

$$\begin{aligned}(p, q) &= \left(\frac{mx_1 - nx_2}{m-n}, \frac{my_1 - ny_2}{m-n}\right) \quad \textit{where} \quad n : m = -1 : 2 \\ &= \left(\frac{2(1) - 1(3)}{2-1}, \frac{2(2) - 1(1)}{2-1}\right) = (-1, 3).\end{aligned}$$

*Thus, the required point is $(-1, 3)$.*

## 1.7 Distance of a Point from a Line

*We consider the distance* $p$ *of* $P(x_1, y_1)$ *from the line* $ax+by+c = 0$*. The distance* $d$ *is given by*

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}} = \pm\frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}}$$

**Proof**

*Consider the illustration in Figure 1.10.*

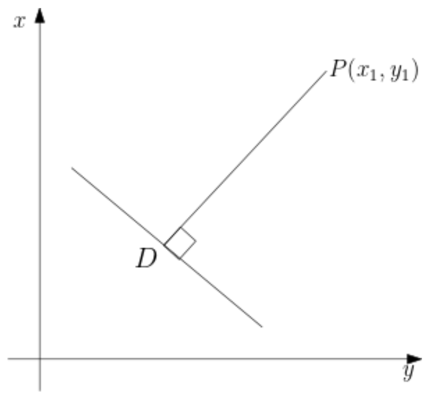



Figure 1.10: Illustration of equation of a point from a line.

*Let* $D$ *denote the point of the perpendicular from* $P$ *to* $ax + by + c = 0$*. Firstly, we want to find the coordinates of the point of the perpendicular* $D$*. However,*

$$\begin{aligned} ax + by + c &= 0 \\ y &= -\frac{a}{b}x - \frac{c}{b}. \end{aligned} \tag{1.7}$$

*The gradient of* $ax + by + c$ *is* $-\dfrac{a}{b}$*. Since* $ax + by + c = 0$ *and* $\overline{DP}$ *are perpendicular, the gradient of*

$$\overline{DP} = -\frac{1}{\left(-\dfrac{a}{b}\right)} = \frac{b}{a}.$$

*Thus, the equation of the line segment $\overline{DP}$ isgiven by*

$$y - y_1 = \frac{b}{a}(x - x_1).$$

*Now to determine the coordinates of $D$, we solve*

$$\begin{aligned} ax + by + c &= 0 \\ y - y_1 &= \frac{b}{a}(x - x_1) \end{aligned} \tag{1.8}$$

*for $x$ and $y$. Thus, from* (1.8)

$$y = \frac{b}{a}(x - x_1) + y_1 \tag{1.9}$$

*and from* (1.7)

$$y = -\frac{a}{b}x - \frac{c}{b}. \tag{1.10}$$

*Equating* (1.9) *and* (1.10) *gives,*

$$\begin{aligned} \frac{b}{a}(x - x_1) + y_1 &= -\frac{a}{b}x - \frac{c}{b} \\ x\left(\frac{b}{a} + \frac{a}{b}\right) &= \frac{b}{a}x_1 - y_1 - \frac{c}{b} \\ x\left(\frac{b^2 + a^2}{ab}\right) &= \frac{b}{a}x_1 - y_1 - \frac{c}{b} \\ x &= \left(\frac{b}{a}x_1 - y_1 - \frac{c}{b}\right)\left(\frac{ab}{b^2 + a^2}\right) \\ x &= \frac{b^2x_1 - aby_1 - ac}{a^2 + b^2}. \end{aligned}$$

*Similarly we obtain,* $y = \dfrac{a^2y_1 - abx_1 - bc}{a^2 + b^2}$*. Therefore, the point $D$ is given by*

$$\left(\frac{b^2x_1 - aby_1 - ac}{a^2 + b^2}, \frac{a^2y_1 - abx_1 - bc}{a^2 + b^2}\right).$$

*Hence,*

$$
\begin{aligned}
|PD|^2 &= \left(x_1 - \frac{b^2x_1 - aby_1 - ac}{a^2+b^2}\right)^2 + \left(y_1 - \frac{a^2y_1 - abx_1 - bc}{a^2+b^2}\right)^2 \\
&= \left(\frac{a^2x_1 + b^2x_1 - b^2x_1 + aby_1 + ac}{a^2+b^2}\right)^2 \\
&+ \left(\frac{a^2y_1 + b^2y_1 - a^2y_1 + abx_1 + bc}{a^2+b^2}\right)^2 \\
&= \left(\frac{a^2x_1 + aby_1 + ac}{a^2+b^2}\right)^2 + \left(\frac{b^2y_1 + abx_1 + bc}{a^2+b^2}\right)^2 \\
&= \frac{a^2(ax_1 + by_1 + c)^2 + b^2(by_1 + ax_1 + c)^2}{(a^2+b^2)^2} \\
&= \frac{(ax_1 + by_1 + c)^2(a^2+b^2)}{(a^2+b^2)^2} \\
&= \frac{(ax_1 + by_1 + c)^2}{a^2+b^2}.
\end{aligned}
$$

*Therefore, the distance of the line segment $|PD|$ is given by*

$$
|PD| = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}} = \pm\frac{(ax_1 + by_1 + c)}{\sqrt{a^2+b^2}}. \qquad (1.11)
$$

***Example 1.7*** *Find the distance of $P(3,2)$ from the line with equation $12x + 5y - 11 = 0$.*

**Solution**

*From the equation $a = 12$, $b = 5$ and $c = -1$. Thus,*

$$
Distance(d) = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}} = \frac{|12(5) + 5(2) - 11}{\sqrt{12^2+5^2}} = \frac{35}{13}.
$$

***Example 1.8*** *Find the set $P$ of points which are at a distance $\sqrt{5}$ from the line $2x - y - 4 = 0$.*

**Solution**

*Let $(h, k)$ be an arbitrary point in the set $P$. Then the distance of $(h, k)$ to the line $2x - y - 4 = 0$ is*

$$\begin{aligned}\sqrt{5} &= \frac{|2h - k - 4|}{\sqrt{2^2 + (-1)^2}} = \pm\frac{(2h - k - 4)}{\sqrt{5}} \\ \pm 5 &= \pm(2h - k - 4)\end{aligned}$$

*Thus,*

$$2h - k - 4 = 5 \qquad \text{and} \qquad 2h - k - 9 = 0,$$

*and*

$$2h - k - 4 = -5 \qquad \text{and} \qquad 2h - k + 1 = 0.$$

*Replacing $h$ and $k$ with $x$ and $y$ respectively gives,*

$$2x - y - 9 = 0 \qquad \text{and} \qquad 2x - y + 1 = 0.$$

*Thus, $P$ consist of the set of points from the lines*

$$2x - y - 9 = 0 \qquad \text{and} \qquad 2x - y + 1 = 0.$$

## 1.8 Equations of Bisectors of Angles Between Two Lines

*Consider the lines $L_1$ and $L_2$. Let $B_1$ and $B_2$ denote the bisectors of the angles between $L_1$ and $L_2$. $B_1$ and $B_2$ form the locus of points equidistant from $L_1$ and $L_2$. Also, $B_1$ is perpendicular to $B_2$, as shown in Figure 1.11.*

***Example 1.9*** *Find the equation of the lines which bisect the angles between the lines $x + y + 2 = 0$ and $x + 7y + 26 = 0$.*

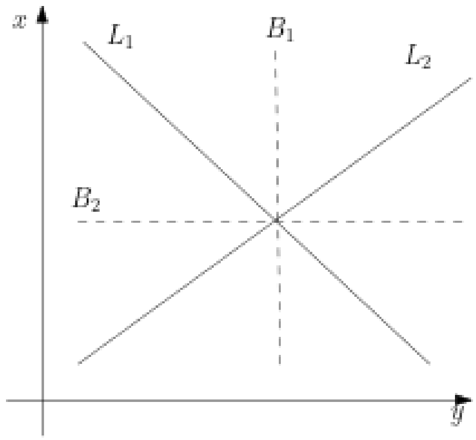



Figure 1.11: Illustration of equation of bisectors of angles between two lines

**Solution**

*Let $(x, y)$ denote any point on any of the bisector. Then $(x, y)$ is equidistant from two lines.*

$$\frac{|x+y+2|}{\sqrt{1^2+1^2}} = \frac{|x+7y+26|}{\sqrt{1^2+7^2}}$$
$$\frac{|x+y+2|}{\sqrt{2}} = \frac{|x+7y+26|}{\sqrt{50}}$$
$$|x+y+2| = \frac{\sqrt{2}}{5\sqrt{2}}|x+7y+26|$$
$$|x+y+2| = \frac{1}{5}|x+7y+26|$$
$$(x+y+2) = \pm\frac{1}{5}(x+7y+26).$$

*Considering the positive part, we have*

$$(x+y+2) = \frac{1}{5}(x+7y+26)$$
$$5x+5y+10 = x+7y+26$$
$$4x-2y-16 = 0.$$

*Finally, the negative part yields*

$$(x+y+2) = -\frac{1}{5}(x+7y+26)$$

$$5x + 5y + 10 = -x - 7y - 26$$
$$6x + 12y + 36 = 0.$$

## 1.9 Exercises

*Students must endeavour to solve all exercises.*

1. *Given A(-3,5) and B(5,-10). Find ;*
   (a) *the distance $AB$*
   (b) *the midpoint $P$ of $AB$*
   (c) *the point $Q$ that divides $AB$ in the ratio 2:5*
   (d) *the point $R$ that divides $AB$ in the ratio 3:1 externally.*
   (e) *the slope of $AB$*
   (f) *the equation of the line $AB$*
   (g) *the equation of the perpendicular bisector of $AB$*
   (h) *the perpendicular distance from S(2,4) to $AB$.*

2. *Find all the values of r such that the slope of the line through the points $(r, 4)$ and $(1, 3 - 2r)$ is less than 5.*

3. *Suppose $M$ is the midpoint of $AB$, where $A$ is (2,3) and $B$ is (18,20). Also, if $P$ divides $AB$ internally and $Q$ divides $AB$ externally in the ratio 2:3. Show that $|MP| \cdot |MQ| = |MB|^2$.*

4. *The equation of two sides of a parallelogram are $y - x = 2$ and $2x + y = 4$. Find the equations of the other sides if they intersect at the point $(0, -4)$.*

5. *Find the gradient of the lines joining the following points;*
   (a) $(cp, \frac{c}{p})$, $(cq, \frac{c}{q})$.
   (b) $(ap^2, 2ap)$, $(aq^2, 2aq)$.
   (c) $(a\cos\theta, b\sin\theta), (a\cos\phi, b\sin\phi)$.

6. *Find the equation of the line with x-intercept 3 and y-intercept -5.*

7. *Find the equation of the line parallel to the x-axis which passes through the point where the lines* $4x + 3y - 6 = 0$ *and* $x - 2y - 7 = 0$ *meet.*

8. *Find the equation of the line passes through the point* $(2, 3)$ *and the point of intersection of the lines* $3x + 2y = 2$ *and* $4x + 3y = 7$.

9. *Given the line* $l$ *with equation* $ax + by + c = 0$ *and the point* $P(x_1, y_1)$.

   (a) *Show that the line through* $P$ *parallel to* $l$ *is given by* $ax + by = ax_1 - ay_1$.

   (b) *Show that the line through* $P$ *perpendicular to* $l$ *is given by* $bx - ay = bx_1 - ay_1$.

10. *If* $A(-2, 1)$, $B(2, 3)$ *and* $C(-2, -4)$ *are three points;*

    (a) *find the angles between the straight lines* $AB$ *and* $BC$.

    (b) *determine whether* $A, B$ *and* $C$ *are collinear.*

11. *At what angle are the lines* $ax + by + c = 0$ *and* $(a - b)x + (a + b)y + d = 0$, $a > 0, b > 0, c > 0$ *inclined to each other?*

12. *One side of a square lies along the straight line* $4x+3y = 26$. *The diagonals of the square intersect at the point* $(-2, 3)$. *Find;*

    (a) *the coordinates of the vertices of the square.*

    (b) *the equation of the sides of the square which are perpendicular to the given line.*

13. *Find the sum of the* $x$ *and* $y$ *intercepts of any tangent line to the curve* $\sqrt{x} + \sqrt{y} = \sqrt{k}$

# Chapter 2

# CIRCLES

*A circle is the locus of a point $P(x, y)$ which moves such that it is at a constant distance from a fixed point. The constant distance is the radius, $r$, and the fixed point is the centre, $c$, as shown in Figure 2.1.*

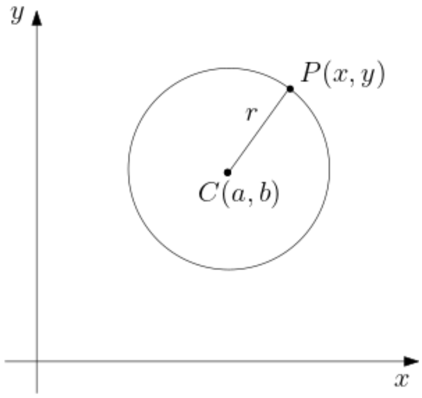



Figure 2.1: Illustration of circle with center $C$ and a point $P$.

*Let $P(x, y)$ be any point on the circle. Then*

$$
\begin{aligned}
|CP| &= r \\
\sqrt{(x-a)^2 + (y-b)^2} &= r \\
|CP|^2 &= r^2 \\
(x-a)^2 + (y-b)^2 &= r^2 \qquad (2.1) \\
x^2 + y^2 - 2ax - 2by &= a^2 + b^2 - r^2 = 0.
\end{aligned}
$$

*Let* $-a = g, -b = f$ *and* $a^2 + b^2 - r^2 = c$,

$$x^2 + y^2 + 2gx + 2fy + c = 0. \tag{2.2}$$

*Thus given an equation of a circle in the form of* (2.2)*, the centre is given by* $(-g, -f)$ *and the radius is*

$$r = \sqrt{a^2 + b^2 - c} = \sqrt{g^2 + f^2 - c}.$$

## 2.1 Properties of the equation of a circle

*The equation of the circle* (2.2) *has perculiar properties as follows;*

1. *The coefficients of* $x^2$ *and* $y^2$ *are equal*
2. *There is no term in* $xy$
3. *The degrees of expressions in* $x$ *and* $y$ *is* 2.

***Example 2.1*** *Find the centre and radius of the circle*

$$x^2 + y^2 + 3x - 6y - 1 = 0. \tag{2.3}$$

**Solution**

*Comparing the equation* (2.3) *with equation* (2.2)*, we have* $2g = 3$ *and* $2f = 6$ *that yields* $g = \frac{3}{2}$ *and* $f = -3$*, respectively. Thus, the centre is given by* $\left(-\frac{3}{2}, 3\right)$*. The radius,* $r$*, is given by*

$$r = \sqrt{\left(\frac{3}{2}\right)^2 + (-3)^2 + 1} = \sqrt{\frac{9}{4} + 10} = \sqrt{\frac{49}{4}} = \frac{7}{2}.$$

## 2.2 Equation of a circle through the origin

*If* $x^2 + y^2 + 2gx + 2fy + c = 0$ *passes through the origin* $O(0, 0)$*, then* $c = 0$*. Thus the equation of the circle through the origin is of the form* $x^2 + y^2 + 2gx + 2fy = 0$.

## 2.3 Equation of a circle on a given diameter

*Let us consider figure 2.2. Suppose* $P_1$ *and* $P_2$ *are the end points*

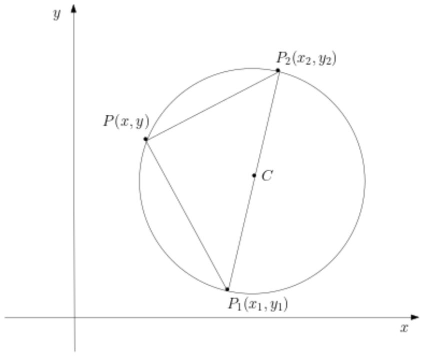



Figure 2.2: Equation of a circle on a diameter.

*of the diameter* $P_1P_2$*. Let* $P(x, y)$ *be any point on the circle such that* $\angle P_1PP_2 = 90^o$*. Thus,* $PP_1$ *is perpendicular to* $PP_2$ *and yields*

$$\left(\frac{y-y_1}{x-x_1}\right)\left(\frac{y-y_2}{x-x_2}\right) = -1$$
$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0.$$

### Method 2

*Find* $C$*, the midpoint of* $P_1P_2$*, we then have the circle with centre* $C$ *and radius* $r =\mid P_1C \mid=\mid CP_2 \mid= \frac{1}{2} \mid P_1P_2 \mid$*.*

***Example 2.2*** *Suppose that* $A(10, 2)$ *and* $B(3, 8)$ *are two points in the xy plane. Find the equation of the circle on* $AB$ *as diameter.*

### Solution

*Let* $P(x, y)$ *be any point on the circle such that* $\angle APB = 90^o$*. Thus,*$AP$ *is perpendicular to* $BP$*.*

$$\left(\frac{y-2}{x-10}\right)\left(\frac{y-8}{x-3}\right) = -1$$
$$(x-10)(x-3)+(y-2)(y-8) = 0$$
$$x^2+y^2-13x-10y+46 = 0.$$

### Method 2

*The Midpoint of AB is* $\left(\frac{10+3}{2}, \frac{2+8}{2}\right) = \left(\frac{13}{2}, 5\right)$. *Thus, the centre of the circle is* $\left(\frac{13}{2}, 5\right)$. *The radius is given by*

$$r = \sqrt{\left(\frac{13}{2}-3\right)^2 + (5-8)^2} = \sqrt{\frac{49}{4}+9} = \sqrt{\frac{85}{4}}$$

*The equation of the circle is*

$$\left(x-\frac{13}{2}\right)^2 + (y-5)^2 = \left(\sqrt{\frac{85}{4}}\right)^2$$
$$\left(x-\frac{13}{2}\right)^2 + (y-5)^2 = \left(\frac{85}{4}\right)$$
$$x^2+y^2-13x-10y+46 = 0.$$

### 2.3.1 Equation of a circle through 3 points

*Consider the circle through 3 points* $P_1(x_1, y_1)$,$P_2(x_2, y_2)$ *and* $P_3(x_3, y_3)$.

### Derivation Steps

1. *Assume the equation of the circle is* $x^2+y^2+2gx+2fy+c = 0$.

2. *Substitute the coordinates of* $P_1$,$P_2$ *and* $P_3$ *in turn, in* (1) *to obtain 3 equations in* $g, f$ *and* $c$.

3. *Solve the three* 3 *equations for* $g, f$ *and* $c$.

4. *Write the equation of the circle.*

***Example 2.3*** *Find the equation of the circle through the points* $(2,1)$,$(0,2)$ *and* $(1,0)$.

**Solution**

*Let the equation of the circle be given by*

$$x^2 + y^2 + 2gx + 2fy + c = 0 \tag{2.4}$$

*Substituting* $(2,1)$ *into* (2.4) *gives*

$$\begin{aligned} 4 + 1 + 4g + 2f + c &= 0 \\ 5 + 4g + 2g + c &= 0. \end{aligned} \tag{2.5}$$

*Also, substituting* $(0,2)$ *into* (2.4) *gives*

$$4 + 4f + c = 0 \tag{2.6}$$

*Finally, substituting* $(1,0)$ *into* (2.4) *gives*

$$1 + 2g + c = 0 \tag{2.7}$$

$(2.6) - (2.5)$ *gives*

$$\begin{aligned} -1 - 4g + 2f &= 0 \\ 4g - 2f &= -1. \end{aligned} \tag{2.8}$$

$(2.5) - (2.7)$ *gives*

$$\begin{aligned} 4 + 2g + 2f &= 0 \\ 2g + 2f &= -4 \end{aligned} \tag{2.9}$$

$(2.8) + (2.9)$ *gives*

$$6g = -5, \quad g = -\frac{5}{6} \qquad \textit{and} \qquad 2f = -\frac{7}{3} \quad \textit{thus} \quad f = -\frac{7}{6}$$

*From* (2.7)*, we obtain*

$$1 - \frac{5}{3} + c = 0 \quad \textit{thus} \quad c = \frac{2}{3}.$$

*Hence, the required equation is*

$$\begin{aligned} x^2 + y^2 + 2\left(\frac{-5}{6}\right)x + 2\left(\frac{-7}{6}\right)y + \frac{2}{3} &= 0 \\ x^2 + y^2 - \frac{5}{3}x - \frac{7}{3}y + \frac{2}{3} &= 0. \end{aligned}$$

## 2.4 Intersection of a circle and a line

*Consider the intersection of the line* $y = mx + c_1$ *and the circle* $x^2 + y^2 + 2gx + 2fy + c_2 = 0$*. In the proceeding diagrams, we illustrate all the possible cases.*

### Case 1

*There are* 2 *distinct points of intersection. Geometrically,* $|CD| < r$ *as shown in Figure 2.3.*

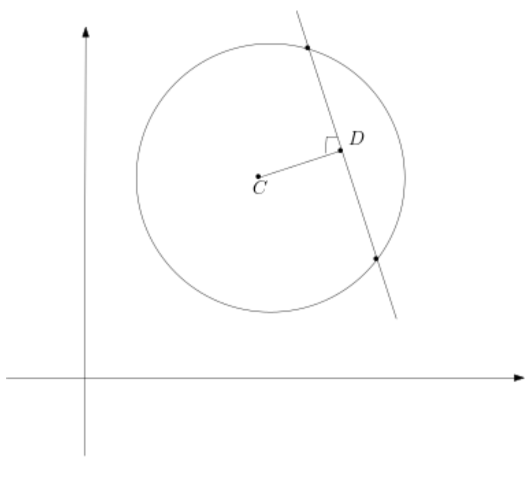



Figure 2.3: Two (2) distinct points of intersection.

### Case 2

*There is one point of intersection. Geometrically, $|CD| = r$ as shown in Figure 2.4.*

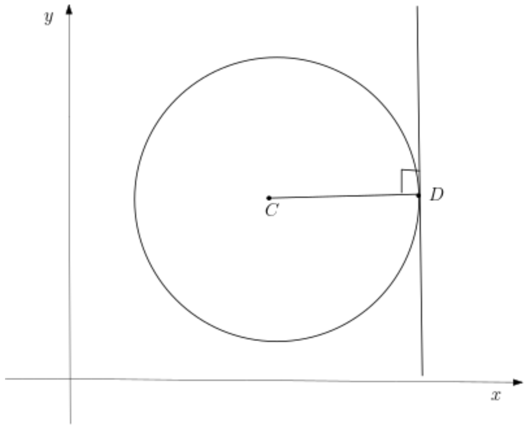



Figure 2.4: One point of intersection.

### Case 3

*There is no point of intersection. Geometrically, $|CD| > r$ as shown in Figure 2.5.*

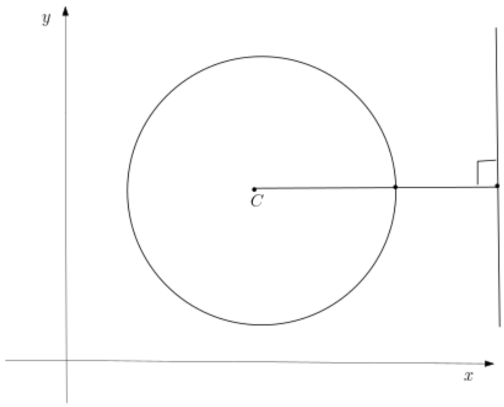



Figure 2.5: No point of intersection.

*Algebraically, let*

$$y = mx + c_1, \tag{2.10}$$

*and*

$$x^2 + y^2 + 2gx + 2fy + c = 0. \qquad (2.11)$$

*Since the points of intersection lie on both the line and the circle, they will be given by the solutions of* (2.10) *and* (2.11) *considered as simultaneous equations in* $x$ *and* $y$. *Substituting* (2.10) *into* (2.11) *gives*

$$x^2 + (mx + c_1)^2 + 2gx + 2f(mx + c_1) + c_2 = 0$$
$$x^2(1 + m^2) + 2x(g + mf + mc_1) + c_1^2 + 2fc_1 + c_2 = 0. \qquad (2.12)$$

*The equation* (2.12) *is a quadratic equation in* $x$ *and will give two values for* $x$, *and from equation* (2.10) *the two corresponding values for* $y$ *are found. The points of intersection will be repeated (i.e, the line will be a tangent) if the roots of equation* (2.12) *are repeated. The condition for this is*

$$(2(g + mf + mc_1)^2)^2 = 4(1 + m^2)(c_2 + 2fc_1 + c_2)$$
$$(2(g + mf + mc_1)^2)^2 = (1 + m^2)(c_2 + 2fc_1 + c_2).$$

*Also, the points of intersection will be distinct if*

$$(2(g + mf + mc_1)^2)^2 > 4(1 + m^2)(c_2 + 2fc_1 + c_2),$$

*moreover, there is no point of intersection if*

$$(2(g + mf + mc_1)^2)^2 < 4(1 + m^2)(c_2 + 2fc_1 + c_2).$$

***Example 2.4*** *Prove that the line* $y = mx + c$ *is tangent to the circle* $x^2 + y^2 = 25$ *if* $c^2 = 25(1 + m^2)$.

**Solution**

*Substituting* $y = mx + c$ *into* $x^2 + y^2 = 25$ *gives,*

$$x^2 + (mx + c)^2 = 25$$
$$x^2 + m^2x^2 + 2mxc + c^2 = 25$$
$$x^2(1 + m^2) + 2mcx + c^2 - 25 = 0.$$

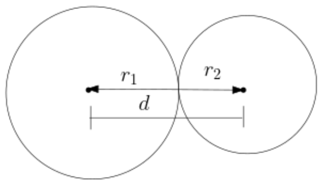
*For tangency,*

$$\begin{aligned}(2mc)^2 - 4(1+m^2)(c^2-25) &= 0\\ 4m^2c^2 - 4(c^2 - 25 + m^2c^2 - 25m^2) &= 0\\ -c^2 + 25 + 25m^2 &= 0\\ c^2 &= 25(1+m^2).\end{aligned}$$

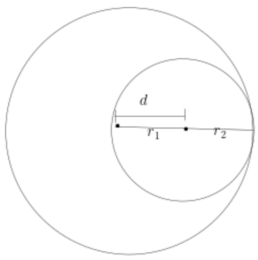
## 2.5 Intersecting Circles

*Consider two circle with radii $r_1$ and $r_2$, $r_1 > r_2$ with centres apart. Then the circles can intersect as follows:*

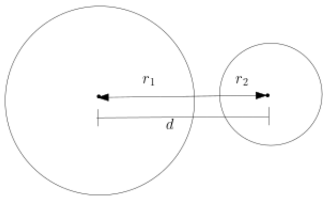
1. *Circles touch externally :* $d = r_1 + r_2$



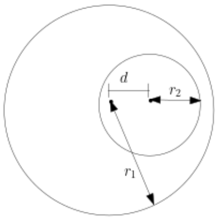

2. *Circles touch internally :* $d = r_1 - r_2$




3. *Circles do not touch each other :* $d > r_1 + r_2$

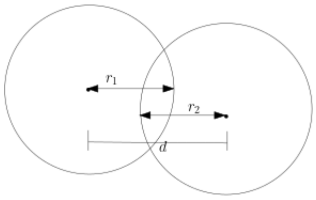



4. *Circles do not touch each other :* $d < r_1 - r_2$

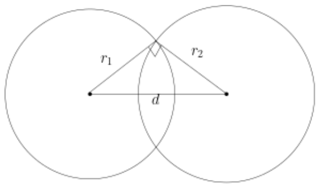



### 2.5.1 Circles intersecting at two distinct points

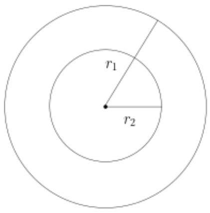
1. $d < r_1 + r_2$




2. $d^2 = r_1^2 + r_2^2$



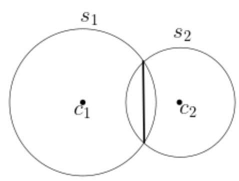

3. $r_1 > r_2$




### 2.5.2 Equation of the common chord

*Given that*

$$s_1 : x^2 + y^2 + 2gx + 2fy + c = 0,$$
$$s_2 : x^2 + y^2 + 2gx + 2fy + c_1 = 0.$$




*The equation of the common chord is given by*

$$S_1 = S_2 \quad or \quad S_1 - S_2 = 0.$$

$$\begin{aligned} S_1 - S_2 &= 0 \\ 2x(g - g_1) + 2y(f - f_1) + (c - c_1) &= 0 \\ ax + by + c &= 0. \end{aligned}$$

*Finally, we have* $a = 2(g - g_1)$, $b = 2(f - f_1)$ *and* $c = c - c_1$.

## 2.6 Exercises

1. *A circle has centre* $(1, 2)$ *and radius 5.*
   (a) *Find the perpendicular distance from the centre of the circle to the line with equation* $x + 2y - 10 = 0$ *and hence show that this line is a tangent to the circle.*
   (b) *Find the perpendicular distance from the centre of the circle to the line with equation* $x + 2y - 12 = 0$ *and hence show that the line does not meet the circle.*
2. *For what value of* $k$ *is the point* $(k, 2k)$ *on the circle with equation* $x^2 + y^2 = 5$*?*
3. *Is the point* $(3, 5)$ *inside, outside or on the circle with equation* $x^2 + y^2 = 9$*?*
4. *For what value of* $k$ *will the line with equation* $x = 6$ *be tangent to the circle with equation* $x^2 + y^2 = k$*?*
5. *Find the equation of the circle that passes through the origin and has intercepts equal to 1 and 2 on the x- and y-axis respectively.*

6. *Find the equation of the tangent at the point* $(0, 2)$ *to the circle:* $x^2 + y^2 - 4x + 2y = 0$

7. *Find the equation of the circle that passes through the point* $(0, 6)$, $(0, 0)$ *and* $(8, 0)$.

8. *Find the point of intersection of the circle with equation* $x^2 + y^2 = 4$ *and circle* $(x - 2)^2 + (y - 2)^2 = 4$.

9. *Find the equation of the circle that has a diameter with end points* $(-6, 1)$ *and* $(2, -5)$.

10. *Find the centre and radius of the following circles*

    (a) $x^2 + y^2 + 6x - 10y = 9$
    (b) $x^2 + y^2 + 4y = 0$
    (c) $-x^2 - y^2 + 8x = 0$
    (d) $(-4 - x)^2 + (-y + 11)^2 = 9$
    (e) $(5 - x)^2 + (y - 1)^2 = 4$

# Chapter 3

# The LIMIT OF A FUNCTION

*The limit describes what happens to the values of $f(x)$ of the function as $x$ approaches the number $a$, as opposed to $f(a)$, which gives the value of the function when $x$ is equal to $a$.*

*We say $f(x)$ has the limit $L$ as $x$ approachs the number $a$ provided that $f(x)$ becomes and remains close to $L$ as $x$ becomes close,but not equal to $a$. This is expressed by writing*

$$\lim_{x \to a} f(x) = L. \tag{3.1}$$

*When such a number $L$ exists, we say that $L$ is the limit of $f(x)$ as $x$ approaches $a$, or simply that $L$ is the limit of $f$ at $a$.*

## 3.1 Evaluating Limits

### 3.1.1 Limits of polynomial functions

*If $f$ is a polynomial function and $a$ is a real number, then*

$$\lim_{x \to a} f(x) = f(a). \tag{3.2}$$

***Example 3.1*** *: Evaluate*

$$\lim_{x \to 1} 2x^2 + 1.$$

**Solution**

*By using the* (3.2), *we obtain*

$$\lim_{x\to 1} 2x^2 + 1 = 2(1)^2 + 1 = 2 + 1 = 3.$$

### 3.1.2 Limits of rational functions

1. *If $q$ is a rational function and $a$ is in the domain of $q$, then*

$$\lim_{x\to a} q(x) = q(a). \tag{3.3}$$

2. *If $q$ is a rational function and $a$ is not in the domain of $q$, then simplify the rational function before evaluating the limit.*

***Example 3.2*** *: Evaluate the following limits.*

*i* $\lim_{x\to 3} \dfrac{x^2 - 2x + 1}{3x - 2}$.

**Solution**

*Since* 3 *is not in the domain of the rational function*

$$\lim_{x\to 3} \frac{x^2 - 2x + 1}{3x - 2} = \frac{(3)^2 - 2(3) + 1}{3(3) - 2} = \frac{4}{7}.$$

*ii* $\lim_{x\to 2} \dfrac{x^2 - 16}{x - 4}$.

**Solution**

*Since* 4 *is not in the domain of the rational function, we will simplify the rational function*

$$\lim_{x\to 4} \frac{x^2 - 16}{x - 4} = \lim_{x\to 4} \frac{(x-4)(x+4)}{x - 4} = \lim_{x\to 4}(x + 4) = 8.$$

## 3.2 Laws of Limits

*If $f$ and $g$ are functions such that $\lim\limits_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exists, then*

i. *The limit of a sum or difference is equal to the sum of the limits.*

$$\lim_{x\to a}\Big[f(x) \pm g(x)\Big] = \Big[\lim_{x\to a} f(x)\Big] \pm \Big[\lim_{x\to a} g(x)\Big].$$

ii. *The limit of a product is equal to the product of the limits.*

$$\lim_{x\to a}\Big[f(x)g(x)\Big] = \Big[\lim_{x\to a} f(x)\Big]\Big[\lim_{x\to a} g(x)\Big].$$

iii. *The limit of a quotient is equal to the quotient of the limits.*

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}.$$

iv. *The limit of a constant times a function is equal to the constant times the limit of the function.*

$$\lim_{x\to a}\Big[cf(x)\Big] = c\lim_{x\to a} f(x).$$

v. *The limit of the $n$-th root of a function is the $n-th$ root of the limit of a function, where $n$ is a positive integer.*

$$\lim_{x\to a}\Big[f(x)\Big]^{1/n} = \left[\lim_{x\to a} f(x)\right]^{1/n}.$$

## 3.3 Limits of trigonometric functions

*To use trigonometric functions, we must understand how to measure* ***angles****. Although we can use both radians and degrees,* ***radians*** *are a more natural measurement because they are related directly to the unit circle, i.e. a circle with radius* 1.

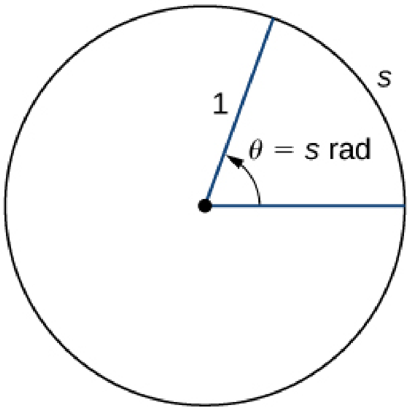



Figure 3.1: The radian measure of an angle $\theta$ is the arc length $s$ of the associated arc on the unit circle.

*The radian measure of an angle is defined as; Given an angle* $\theta$*, let* $s$ *be the length of the corresponding arc on the unit circle Figure 3.1. We say the angle corresponding to the arc of length* 1 *has radian measure* 1.

*The basic trigonometric limit is given by the following theorem*

***Theorem 3.1*** *Let* $x$ *be an angle measured in radians. Then:*

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

***Example 3.3*** *Evaluate*

$$\lim_{x\to 0}\frac{\tan x}{x} = 1.$$

**Solution**

*Using trigonometric identities, we have*

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x}{x\cos x} &= \lim_{x\to 0}\left[\frac{\sin x}{x}\right]\left[\frac{1}{\cos x}\right]\\ &= \left[\lim_{x\to 0}\frac{\sin x}{x}\right]\left[\lim_{x\to 0}\frac{1}{\cos x}\right] = [1]\left[\frac{1}{\cos 0}\right] = 1.\end{aligned}$$

***Example 3.4*** *Evaluate* $\lim_{x\to 0}\dfrac{\sin 4x}{x}$.

**Solution**

*Using Theorem 3.1, we obtain*

$$\lim_{x\to0}\frac{\sin 4x}{x}=\lim_{x\to0}\frac{4\sin 4x}{4x}=4\lim_{x\to0}\frac{\sin 4x}{4x}=4(1)=4.$$

***Example 3.5*** *Evaluate*

$$\lim_{x\to0}\frac{\cos 3x-\cos x}{x^2}.$$

**Solution**

*We factor the numerator as follows*

$$\cos 3x-\cos x=-2\sin\frac{3x-x}{2}\sin\frac{3x+x}{2}=-2\sin x\sin 2x.$$

*This yields*

$$\lim_{x\to0}\frac{\cos 3x-\cos x}{x^2}=\lim_{x\to0}\frac{(-2\sin x\sin 2x)}{x^2}=-2\lim_{x\to0}\frac{\sin x}{x}\lim_{x\to0}\frac{\sin 2x}{x}$$
$$=-2\cdot1\cdot\lim_{x\to0}\frac{2\sin 2x}{2x}=-2\cdot1\cdot2\cdot\lim_{x\to0}\frac{\sin 2x}{2x}=-4.$$

## 3.4 One-Sided Limits

*If* $\lim_{x\to a^+} f(x)=L$, *we say that the limit of* $f(x)$ *as* $x$ *approaches* $a$ *from the right is* $L$. *We also refer to* $L$ *as the right-hand limit of* $f(x)$ *as* $x$ *approaches* $a$. *The symbol* $x\to a^+$ *is used to indicate that* $x$ *is restricted to values greater than* $a$.

*Also, if* $\lim_{x\to a^-} f(x)=L$, *we say that the limit of* $f(x)$ *as* $x$ *approaches* $a$ *from the left is* $L$, *or that* $L$ *is the left side limit of* $f(x)$ *as* $x$ *approaches* $a$. *The symbol* $x\to a^-$ *is used to indicate that* $x$ *is restricted to values less than* $a$.

***Example 3.6*** *Let* $f(x)=|x|$. *Find*

*a.* $\lim_{x\to0^+} f(x)$

*b.* $\lim_{x\to0^-} f(x)$.

**Solution**

*First, we rewrite* $f(x) = |x|$ *as*

$$f(x) = \begin{cases} x & if \quad x \geq 0, \\ -x & if \quad x < 0. \end{cases}$$

*Then, we evaluate the limits as follows*

*a.* $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} x = 0.$

*b.* $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (-x) = 0.$

## 3.5 Relationship between one-sided limits and two-sided limits.

*The limit of a function i.e.* $\lim\limits_{x \to a} f(x) = L$ *if and only if both* $\lim\limits_{x \to a^-} f(x) = L$ *and* $\lim\limits_{x \to a^+} f(x) = L$.

***Example 3.7*** *Evaluate*

$$\lim_{x \to 8} \frac{|x-8|}{x-8}.$$

**Solution**

*We rewrite* $\lim\limits_{x \to 8} \dfrac{|x-8|}{x-8}$ *as*

$$f(x) = \begin{cases} \dfrac{x-8}{x-8} & if \quad x > 0, \\ -\dfrac{(x-8)}{x-8} & if \quad x < 0. \end{cases}$$

$$f(x) = \begin{cases} 1 & if \quad x > 0, \\ -1 & if \quad x < 0. \end{cases}$$

*Therefore*

$$\lim_{x\to 8^+} \frac{|x-8|}{x-8} = \lim_{x\to 8^+} 1 = 1, \quad \textit{and} \quad \lim_{x\to 8^-} \frac{|x-8|}{x-8} = \lim_{x\to 8^-} (-1) = -1.$$

*Since* $\lim_{x\to 8^+} \frac{|x-8|}{x-8} \neq \lim_{x\to 8^-} \frac{|x-8|}{x-8}$, *the* $\lim_{x\to 8} \frac{|x-8|}{x-8}$ *does not exist.*

## 3.6 The Sandwich (Squeeze) Theorem

*Suppose that* $f$,$g$ *and* $h$ *are functions with*

$$f(x) \leqslant g(x) \leqslant h(x),$$

*for each* $x \neq a$ *in an open interval containing* $a$. *If*

$$\lim_{x\to a} f(x) = L \quad \textit{and} \quad \lim_{x\to a} h(x) = L,$$

*then* $\lim_{x\to a} g(x) = L$.

***Example 3.8*** *Use the sandwich (squeeze) theorem to evaluate* $\lim_{x\to 0} \frac{x^2 \cos x}{x^2+1}$.

**Solution**

*Since* $-1 \leq \cos x \leq 1$ *for all* $x$

$$\frac{-x^2}{x^2+1} \leq \frac{x^2 \cos x}{x^2+1} \leq \frac{x^2}{x^2+1}.$$

*However,*

$$\lim_{x\to 0} \frac{-x^2}{x^2+1} = 0 \quad \textit{and} \quad \lim_{x\to 0} \frac{x^2}{x^2+1} = 0,$$

*so by the sandwich theorem, we obtain*

$$\lim_{x\to 0} \frac{x^2 \cos x}{x^2+1} = 0.$$

## 3.7 Continuity

*The common usage of continuous signifies behaviour without break. The term continuous is used in calculus to describe functions whose graphs have this type of behaviour. Consider the following graphs*

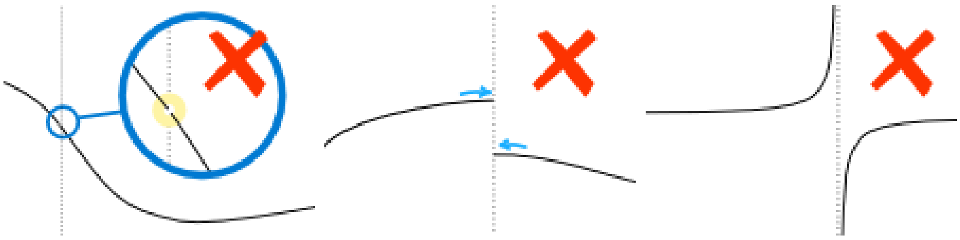


Figure 3.2: Examples of not continuous (discontinuous) functions: (a). Holes (b). Jumps and (c). Vertical asymptotes.

***Definition 3.1*** *A function $f$ is continuous at $a$ if*

*i.* $f(a)$ *exists, that is $a$ is in the domain of $f$;*

*ii.* $\lim_{x\to a} f(x)$ *exists; and*

*iii.* $\lim_{x\to a} f(x) = f(a)$.

*If a function is not continuous at $a$, then $f$ is said to be discontinuous at $a$.*

***Example 3.9*** *Determine whether* $f(x) = \dfrac{x^2-4}{x^2-3x+2}$ *is continuous at $a = 2$.*

**Solution**

*From the first definition of continuity, we have*

$$f(2) = \frac{2^2-4}{2^2-6+2} = \frac{0}{0}.$$

*Thus $f(2)$ is undefined and therefore $f(x)$ is not continuous at $a = 2$.*

***Example 3.10*** *Find the constant c that will make*

$$f(x) = \begin{cases} \dfrac{x^2-1}{x-1} & if \quad x \neq 1, \\ c & if \quad x = 1, \end{cases}$$

*continuous at $x = 1$.*

**Solution**

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{(x-1)} = \lim_{x\to 1}(x+1) = 2.$$

*But for $f(x)$ to be continuous, we need $\lim_{x\to 1} f(x) = f(1) = 2$. Therefore $c = 2$.*

## 3.8 Exercises

*Students must endeavour to solve all exercises.*

1. *Find the following limits, if they exist.*

   *i.* $\lim_{x\to 1} \dfrac{x^4-1}{x-1}$

   *ii.* $\lim_{x\to 0} \dfrac{2-\cos 3x - \cos 4x}{x}$

   *iii.* $\lim_{x\to 8} \dfrac{x-8}{\sqrt[3]{x}-2}$

   *iv.* $\lim_{\tau\to 0} \dfrac{\tau^2}{1-\cos^2\tau}$

   *v.* $\lim_{x\to 1} \dfrac{x|x-1|}{x-1}$

   *vi.* $\lim_{x\to 0} \dfrac{1-\cos 4\theta}{1-\cos 6\theta}$

   *vii.* $\lim_{t\to 0} \dfrac{\sqrt{2+t}-\sqrt{2}}{t}$

   *viii.* $\lim_{x\to 0} \dfrac{\sin(\sin x)}{x}$

   *ix.* $\lim_{x\to 1.5} \dfrac{2x^2-3x}{|2x-3|}$

   *x.* $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta-\tan\theta}$

   *xi.* $\lim_{x\to 2} \dfrac{|x-2|}{x^3-8}$

   *xii.* $\lim_{x\to 1} \cos\left(\dfrac{x^2-1}{x-1}\right)$

*xiii.* $\lim_{x\to 0} \frac{1-\cos x}{x}$

*xiv.* $\lim_{x\to 0} \frac{x^2 - 3\sin x}{x}$

*xv.* $\lim_{x\to 4} |10 - 3x^2|$

*xvi.* $\lim_{x\to 0} \frac{x}{\cos\left(\frac{1}{2}\pi - x\right)}$

2. *Given*

$$f(x) = \begin{cases} \frac{1}{x+2}, & x < -2, \\ x^2 - 5, & -2 \le x \le 3, \\ \sqrt{x+13}, & x > 3. \end{cases}$$

*Find*

*i* $\lim_{x\to -2^+} f(x)$

*ii* $\lim_{x\to -2^-} f(x)$

*iii* $\lim_{x\to -2} f(x)$

*iv* $\lim_{x\to 3} f(x)$

3. *If*

$$f(x) = \begin{cases} -x-2 & if \quad x \le -1 \\ x & if \quad -1 < x < 1 \\ x^2 - 2x & if \quad x \ge 1 \end{cases}$$

*determine whether or not* $\lim_{x\to -1} f(x)$ *and* $\lim_{x\to 1} f(x)$ *exist.*

4. *Find the derivative of* $f(x) = ax^2 + bx + c$, *where* $a, b$ *and* $c$ *are none-zero constant, by limit definition.*

5. *Find the derivative of the following functions by first principle.*

   *(a)* $f(x) = (x-1)(x-2)$

   *(b)* $f(t) = \frac{4t}{t+1}$

   *(c)* $f(x) = \sin x$

   *Hint:* $\sin(x+h) - \sin x = 2\cos\left(\frac{(x+h)+x}{2}\right)\sin\left(\frac{(x+h)-x}{2}\right)$

(d) $f(x) = \tan(ax + b)$

6. For what value of the constant c is the function

$$f(x) = \begin{cases} x + c & if \quad x \neq 2 \\ cx^2 + 1 & if \quad x = 2 \end{cases}$$

continuous at every number.

7. Find the $\lim_{x \to -1}$ if: $\frac{1}{2} \leq \frac{f(x)}{2} \leq \frac{x^2 + 2x + 2}{2}$.

8. If $\lim_{x \to 4} f(x) = 4$, evaluate $\lim_{x \to 4} \sqrt{f(x) + 3x}$.

9. Find the value of a so that $\lim_{x \to 1} f(x)$ exist when

$$f(x) = \begin{cases} 3x + 5, & x \leq 1 \\ 2x + a, & x > 1 \end{cases}$$

10. Evaluate $\lim_{x \to 3} \frac{\sqrt{12 - x} - x}{\sqrt{6 + 3} - 3}$.

11. Evaluate the following limits.

(a) $\lim_{x \to \frac{\pi}{2}} \frac{1 + \cos 2x}{\frac{\pi}{2}(\pi - 2x)^2}$

(b) $\lim_{x \to 0} \frac{x + 2\sin x}{\sqrt{x^2 + 2\sin x + 1} - \sqrt{\sin^2 x - x + 1}}$

12. For what value of a and b will

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}; & x \leq 2, \\ ax^2 - bx + 3; & 2 \leq x \leq 3, \\ 2x - a + b; & x \geq 3, \end{cases}$$

be continuous at $\mathbb{R}$.

# Chapter 4

# THE DERIVATIVE OF A FUNCTION

*We all know how to find the slope of a straight line. You simply divide the change in $y$ by the change in $x$. This is commonly known as the* `rate of change`*. Slopes of linear equations are constant across the entire line. However, if we consider a curve, there won't be a constant slope for the entire function. In such instances, we seek to find an equation that we can use to give us the slope of a line tangent to the curve at any given value of $x$. Using this equation gives us the instantaneous rate of change or the slope at a specific point on the curve.*

***Definition 4.1*** *The derivative of a function is a function $f'$ defined by*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{4.1}$$

***Example 4.1*** *Use the definition of the derivative to find $f'(x)$ if $f(x) = 4x - 2$.*

**Solution**

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{4(x+h) - 2 - (4x - 2)}{h} \\ &= \lim_{h \to 0} \frac{4x + 4h - 2 - 4x + 2}{h} = \lim_{h \to 0} \frac{4h}{h} = \lim_{h \to 0} 4 = 4. \end{aligned}$$

***Example 4.2*** *Using the first principle, find $f'(x)$ if $f(x) = \dfrac{1}{x^2}$.*

**Solution**

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h} = \lim_{h\to 0} \frac{x^2 - x^2 - 2xh - h^2}{x^2(x+h)^2 h} \\
&= \lim_{h\to 0} \frac{x^2 - (x+h)^2}{x^2(x+h)^2 h} = \lim_{h\to 0} \frac{-h(2x+h)}{x^2(x+h)^2 h} = \lim_{h\to 0} \frac{-2x-h}{x^2(x+h)^2} \\
&= \frac{-2x}{x^4} = \frac{-2}{x^3}.
\end{aligned}
$$

*The notation for the derivative can also assume different forms. Common alternative notations for $f'(x)$ are*

$$\frac{d}{dx} f(x), D_x f(x).$$

*Also, when $f$ is defined by $y = f(x)$, it is also common to use $\dfrac{dy}{dx}$ and $y'$ to denote the derivative.*

## 4.1 Rules for differentiation

### 4.1.1 The derivative of a constant function

*If $f(x) = c$, where $c$ is a constant, then $f'(x) = 0$.*

***Example 4.3*** *Find $f'(x)$ if $f(x) = 200\pi$.*

### 4.1.2 The Power Rule

*If $f(x) = x^n$, where $n$ is any real number, then*

$$f'(x) = nx^{n-1}. \tag{4.2}$$

***Example 4.4*** *If $f(x) = x^5$, find $f'(x)$.*

**Solution**

*By the power rule* $f'(x) = 5x^4$.

***Example 4.5*** *If* $f(x) = \sqrt{x}$, *find* $f'(x)$.

**Solution**

*First, we rewrite the function* $f(x) = \sqrt{x} = x^{1/2}$. *Then, we have*

$$f'(x) = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2}.$$

***Example 4.6*** *If* $f(x) = \dfrac{1}{\sqrt[3]{x}}$, *find* $f'(x)$.

**Solution**

*We rewrite the function as follows* $f(x) = \dfrac{1}{\sqrt[3]{x}} = x^{-1/3}$. *Then, we have*

$$f'(x) = -\frac{1}{3}x^{-1/3-1} = -\frac{1}{3}x^{-4/3}.$$

### 4.1.3 The Sum Rule

*If* $f$ *and* $g$ *are differentiable at* $x$, *then*

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x). \tag{4.3}$$

***Example 4.7*** *Find* $f'(x)$ *if* $f(x) = x^7 + \sqrt{x}$.

**Solution**

*Firstly, we apply the sum rule* (4.3) *then followed by the power rule* (4.2) *as follows*

$$\frac{d}{dx}f(x) = \frac{d}{dx}[x^7 + \sqrt{x}] = \frac{d}{dx}[x^7] + \frac{d}{dx}[x^{1/2}] = 7x^6 + \frac{1}{2}x^{-1/2}.$$

***Remark 4.1*** *If $f$ is differentiable at $x$, then for any constant $c$,*

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)].$$

***Example 4.8*** *Find $f'(x)$ for the following functions.*

1. $f(x) = 4x^5$

   **Solution**

   *Using Remark 4.1, we obtain*

   $$\frac{d}{dx}[4x^5] = 4\frac{d}{dx}[x^5] = 4[5x^4] = 20x^4.$$

2. $f(x) = 3\sqrt{x} + 5x^2$

   **Solution**

   *We proceed as follows*

   $$\begin{aligned}\frac{d}{dx}[3\sqrt{x} + 5x^2] &= \frac{d}{dx}(3(x^{1/2})) + \frac{d}{dx}(5(x^2)) \\ &= 3\left[\frac{1}{2}x^{-1/2}\right] + 5(2)x = \frac{3}{2}x^{-1/2} + 10x.\end{aligned}$$

### 4.1.4 The Product Rule

*If $f$ is differentiable at $x$, then*

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]. \tag{4.4}$$

***Example 4.9*** *Calculate $\dfrac{dy}{dx}$ if $y = (x^2 + x - 1)(2x + 4)$.*

## Solution

$$\frac{dy}{dx} = (x^2 + x - 1)(2) + (2x + 4)(2x + 1) = 6x^2 + 12x + 2.$$

#### The Quotient Rule

*If $f$ and $g$ is differentiable at $x$ and $g(x) \neq 0$, then*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}. \qquad (4.5)$$

***Example 4.10*** *Find $f'(x)$, if $f(x) = \dfrac{x^4 + 2x - 1}{3x^2 + 5}$.*

## Solution

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\frac{x^4 + 2x - 1}{3x^2 + 5}\right] \\ &= \frac{(3x^2 + 5)\frac{d}{dx}(x^4 + 2x - 1) - (x^4 + 2x - 1)\frac{d}{dx}(3x^2 + 5)}{[3x^2 + 5]^2} \\ &= \frac{(3x^2 + 5)(4x^3 + 2) - (x^4 + 2x - 1)(6x)}{[3x^2 + 5]^2}. \end{aligned}$$

### 4.1.5 Differentiation of trigonometric functions

***Theorem 4.2*** *For any real number $x$,*

$$\frac{d}{dx}[\sin x] = \cos x \quad \textit{and} \quad \frac{d}{dx}[\cos x] = -\sin x. \qquad (4.6)$$

***Example 4.11*** *Let $f(x) = x \sin x - \cos x$. Find $f'(x)$.*

**Solution**

*Using the sum rule (4.3), product rule (4.4) and the trigonometric differentiation rule (4.6), we have*

$$\begin{aligned}\frac{d}{dx}[x\sin x-\cos x] &= \frac{d}{dx}[x\sin x]-\frac{d}{dx}[\cos x]\\ &= x\frac{d}{dx}[\sin x]+\sin x\frac{d}{dx}[x]-\frac{d}{dx}[\cos x]\\ &= x\cos x+\sin x+\sin x = x\cos x+2\sin x.\end{aligned}$$

***Example 4.12*** *Find* $\frac{dy}{dx}$ *if* $y=\frac{x^2+2}{1+\sin x}$.

**Solution**

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1+\sin x)\frac{d}{dx}(x^2+2)-(x^2+2)\frac{d}{dx}(1+\sin x)}{[1+\sin x]^2}\\ &= \frac{(1+\sin x)(2x)-(x^2+2)(\cos x)}{[1+\sin x]^2}\\ &= \frac{2x+2x\sin x-x^2\cos x-2\cos x}{[1+\sin x]^2}\end{aligned}$$

***Theorem 4.3*** *For each real number $x$, for which the functions are defined:*

1. $\frac{d}{dx}\tan x=\sec^2 x$
2. $\frac{d}{dx}\sec x=\sec x\tan x$
3. $\frac{d}{dx}\cot x=-\csc^2 x$
4. $\frac{d}{dx}\csc x=-\csc x\cot x.$

***Example 4.13*** *Find* $f'(x)$ *if* $f(x)=\frac{\tan x}{1+2\sec x}$.

**Solution**

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\left[\frac{\tan x}{1+2\sec x}\right] \\
&= \frac{(1+2\sec x)\frac{d}{dx}(\tan x) - \tan x\frac{d}{dx}(1+2\sec x)}{(1+2\sec x)^2} \\
&= \frac{(1+2\sec x)(\sec^2 x) - \tan x(2\sec x\tan x)}{(1+2\sec x)^2}.
\end{aligned}
$$

***Example 4.14*** *Find the derivative of* $f(x) = \cot x$.

**Solution**

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\left(\frac{1}{\tan x}\right) = \left(\frac{\cos x}{\sin x}\right) \\
&= \frac{\sin x\frac{d}{dx}(\cos x) - \cos x\frac{d}{dx}(\sin x)}{\sin^2 x} \\
&= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{sin^2 x} = -\csc^2 x.
\end{aligned}
$$

### 4.1.6 The Chain Rule (Composite functions)

*If* $f$ *is differentiable at* $x$ *and* $f$ *is differentiable at* $g(x)$, *then* $f(g(x))$ *is differentiable at* $x$ *and*

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x) = \frac{d}{dx}[(f\circ g)(x)]$$

*An easy way to remember the chain rule is to suppose that* $u = g(x)$ *and* $y = f(u)$ *then*

$$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx}.$$

***Example 4.15*** *Differentiate* $y = (x^2+x)^{-5}$.

**Solution**

*Let* $u = x^2 + x$, *then* $\dfrac{du}{dx} = 2x + 1$. *Thus* $y = u^{-5}$ *yields* $\dfrac{dy}{du} = -5u^{-6}$. *Therefore,*

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -5u^{-6}(2x+1) = -5(x^2+x)^{-6}(2x+1).$$

***Example 4.16*** *Find* $\dfrac{dy}{dx}$ *if* $y = \sin^5 x$.

**Solution**

*Let* $u = \sin x$ *then* $y = u^5$. *This yields* $\dfrac{du}{dx} = \cos x$ *and* $\dfrac{dy}{du} = 5u^4 = 5\sin^4 x$. *Thus*

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 5sin^4 x \cos x.$$

***Example 4.17*** *Find* $\dfrac{dy}{dx}$ *if* $y = \tan(3x^2 + 1)$.

**Solution**

*Let* $u = 3x^2 + 1$ *then* $y = \tan u$. *Therefore,*

$$\frac{du}{dx} = 6x, \quad \textit{and} \quad \frac{dy}{du} = \sec^2 u = \sec^2(3x^2+1).$$

*Thus*

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \sec^2(3x^2+1)(6x).$$

*Alternatively,*

$$\frac{dy}{dx} = \sec^2(3x^2+1)\frac{d}{dx}\left(3x^2+1\right) = 6x\sec^2(3x^2+1).$$

### 4.1.7 The General Power Rule

*The general power rule is a special case of the chain rule. It is useful when finding the derivative of a function that is raised to the n-th power. The general power rule states that this derivative is n times the function raised to the $(n-1)$-th power times the derivative of the function.*

$$\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}f'(x). \tag{4.7}$$

***Example 4.18*** *Find $y'$ if $y = \cos^3(x^2+1)$.*

**Solution**

*Using the general power rule* (4.7), *we have*

$$\begin{aligned} y' &= 3\cos^2(x^2+1)\frac{d}{dx}[\cos(x^2+1)] \\ &= 3\cos^2(x^2+1)[-\sin(x^2+1)(2x)] \\ &= -6cos^2(x^2+1)\sin(x^2+1). \end{aligned}$$

## 4.2 Implicit Differentiation

*The functions that we have met so far can be described by expressing one variable explicitly be described in terms of another variable- for example, $y = \sqrt{x^3+1}$ or $y = x\sin x$, or in general $y = f(x)$. Some functions, however, are defined implicitly by a relation between $x$ and $y$ such as $x^2+y^2 = 25$ or $\sin(xy) = 4$.*

*Fortunately, we don't need to solve an equation for $y$ in terms of $x$ in order to find the derivative of $y$. Instead, we can use the method of implicit differentiation. This consists of differentiating both sides of the equation with respect to $x$ and then solving the resulting equation for $y'$.*

***Example 4.19*** *Use implicit differentiation to find $\frac{dy}{dx}$ if $xy^2 + 6y + x = 0$.*

**Solution**

$$\begin{aligned}
\frac{d}{dx}[xy^2 + 6y + x] &= \frac{d}{dx}[0] \\
y^2 2xyy' + 6y' + 1 &= 0 \\
y'[2xy + 6] &= -1 - y^2 \\
y' &= \frac{-1 - y^2}{2xy + 6}.
\end{aligned}$$

***Example 4.20*** *Find $y'$ if $\cos(xy) + \sin x = 1$.*

**Solution**

$$\begin{aligned}
\frac{d}{dx}\left[\cos(xy) + \sin x\right] &= \frac{d}{dx}[1] \\
-\sin(xy)[y + xy'] + \cos x &= 0 \\
y' &= \frac{y\sin(xy) - \cos x}{-x\sin(xy)} = \frac{\cos x - y\sin(xy)}{x\sin(xy)}.
\end{aligned}$$

## 4.3 Differentiation of the Natural Exponential Function

**Rule 1**

$$\frac{d}{dx}\left[e^x\right] = e^x.$$

**Rule 2**

*If $g(x)$ is differentiable, then*

$$\frac{d}{dx}\left[e^{g(x)}\right] = g'(x)e^{g(x)}$$

***Example 4.21*** *Find $\frac{dy}{dx}$ if $y = e^{x^2}$.*

**Solution**

$$\frac{dy}{dx} = \frac{d}{dx}[x^2]e^{x^2} = 2xe^{x^2}.$$

***Example 4.22*** *If* $y = \exp^x \sin(2x) + e^{\cos x}$, *find* $y'$.

**Solution**

$$y' = e^x \sin(2x) + e^x \cos(2x)(2) - \sin x e^{\cos x}.$$

## 4.4 Differentiation of the Natural Logarithmic Function

**Rule 1**

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

**Rule 2**

*If* $g(x)$ *is differentiable at* $x$, *then*

$$\frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$$

***Example 4.23*** 1. *Find* $y'$ *if* $y = \ln(\sin x)$.

**Solution**

$$y' = \frac{\frac{d}{dx}[\sin x]}{\sin x} = \frac{\cos x}{\sin x} = \cot x.$$

2. *Find* $y'$ *if* $y = \sin(x^2+1)\ln x + \ln(4x^3+3)$.

**Solution**

$$y' = \frac{1}{x}\sin(x^2+1) + \ln x \cos(x^2+1)(2x) + \frac{12x^2}{4x^3+3}.$$

## 4.5 Differentiation of General Exponential and Logarithmic Functions

**Rule 1**

*Let $y = a^x$, then*

$$y = e^{\ln a^x} = e^{x \ln a}.$$

*This implies*

$$y = \ln a e^{x \ln a},$$

*and finally,*

$$y' = (\ln a)a^x.$$

***Example 4.24*** *Find $\frac{dy}{dx}$ if $y = 4^x$.*

**Solution**

$$y' = \ln 4(4^x).$$

**Rule 2**

$\frac{d}{dx}[a^{g(x)}] = [\ln a][a^{g(x)}]g'(x)$ *i.e.* $e^{\ln a^{g(x)}} = e^{g(x)\ln a}$ *implies*

$$g'(x) \ln a e^{g(x) \ln a} = [\ln a][a^{g(x)}]g'(x).$$

***Example 4.25*** *Find $y'$ if $y = 6^{\sin x}$.*

**Solution**

$y' = (\ln 6)6^{\sin x}\cos x.$

***Example 4.26*** *Find $y'$ if $y = (x^2+1)^{10} + 10^{x^2+1}$.*

## Solution

$$\begin{aligned} y' &= 10(x^2+1)^9(2x) + (\ln 10)10^{x^2+1}(2x) \\ &= 20x(x^2+1)^9 + (\ln 10)10^{x^2+1}(2x) \end{aligned}$$

## Rule 3

*Let*

$$y = \log_a x \quad \textit{then} \quad a^y = x.$$

*Therefore,*

$$\ln a^y = \ln x \quad \textit{then} \quad y = \frac{\ln x}{\ln a}.$$

*This implies*

$$\frac{dy}{dx} = \frac{1}{x \ln a}.$$

***Example 4.27*** *Find $y'$ if $y = \log_4 x$.*

## Solution

$$y' = \frac{1}{x \ln 4}$$

## Rule 4

*Let*

$$y = \log_a |g(x)| \quad \textit{then} \quad \frac{dy}{dx} = \frac{1}{g(x) \ln a} \frac{d}{dx}[g(x)].$$

***Example 4.28*** *Find $f'(x)$ if $f(x) = \log \sqrt[3]{(2x+5)^2}$.*

## Solution

$$f(x) = \log(2x+5)^{2/3} = \frac{2}{3} \log |2x+5|.$$

*Thus,*

$$f'(x) = \frac{2}{3} \frac{1}{(2x+5)\ln 1}(2x) = \frac{4}{(6x+15)\ln 10}.$$

## 4.6 Logarithmic Differentiation

*This method is used to differentiate functions of the form* $f(x)^{g(x)}$.

***Example 4.29*** *Find* $y'$ *if* $y = 3^x$.

**Solution**

*If* $y = 3^x$, *then* $\ln y = \ln 3^x$.

$$\begin{aligned} \ln y &= x \ln 3 \\ \frac{1}{y} y' &= \ln 3 \\ y' &= \ln 3 y = \ln 3(3^x). \end{aligned}$$

***Example 4.30*** *Find* $y'$ *if* $y = x^{\sin x}$.

**Solution**

*If* $\ln y = \ln x^{\sin x}$

$$\begin{aligned} \frac{1}{y} y' &= \cos x \ln x + \sin x (\frac{1}{x}) \\ y' &= \left[ \cos x \ln x + \left( \frac{1}{x} \right) \sin x \right] x^{\sin x}. \end{aligned}$$

## 4.7 Higher Order Derivatives

*The derivative of a function* $f$ *leads to another function* $f'$ *if* $f'$ *has a derivative, it is denoted by* $f''$ *and is called the* ***second derivative*** *of* $f$. *Also, the third derivative* $f'''$ *of* $f$ *is the derivative of the second derivative.*

*If* $y = f(x)$, *then the first* $n$ *derivatives are denoted by*

$$y', y'', y''', \ldots, y^{(n)}.$$

*If the differential notation is used, we write*

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots, \frac{d^ny}{dx^n}.$$

***Example 4.31*** *Given* $2x + 2yy' = 0$, *show that* $y''y^3 + 1 = 0$.

**Solution**

*The first derivative is* $y' = -\dfrac{x}{y}$. *Then,*

$$\begin{aligned} y'' = \frac{-[y - xy']}{y^2} &= \frac{-[y - x\left(-\frac{x}{y}\right)]}{y^2} \\ &= -\frac{y + x^2/y}{y^2} \\ &= -\frac{y^2 + x^2}{y.y^2} = -\frac{1}{y^3}. \end{aligned}$$

## 4.8 Tangents and Normal

**Tangent**

*A tangent to a curve is a line that touches the curve at one point and has the same slope as the curve at that point, see, e.g. Figure 4.1.*

*Consider the tangent to the curve* $y = f(x)$ *at* $A(x_1, y_1)$. *Then gradient of tangent=* $f'(x_1)$. *The equation of tangent is given by*

$$y - y_1 = f'(x_1)(x - x_1). \tag{4.8}$$

***Example 4.32*** *Find the equation of the tangent at* $(1, 2)$ *on the curve* $y = x^2 + 4x - 3$.

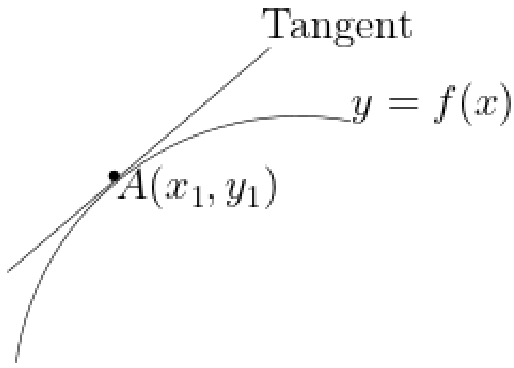



Figure 4.1: Illustation of tangent line.

## Solution

*Let* $y' = 2x + 4$, *then the gradient of the tangent at* $(1, 2)$ *is given by*

$$f'(1) = 2(1) + 4 = 6.$$

*The equation of the tangent is*

$$y - 2 = 6(x - 1) \quad hence \quad y = 6x - 4.$$

#### Normal

*A normal to a curve is a line perpendicular to a tangent to the curve, see e.g. Figure 4.2.*

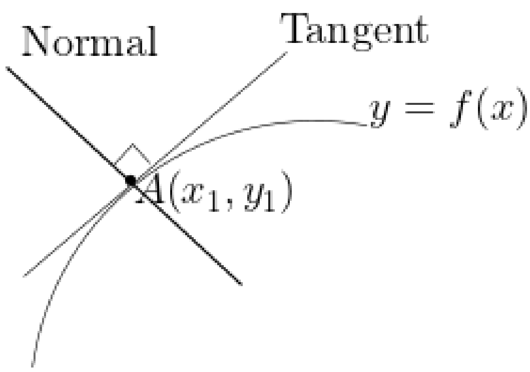



Figure 4.2: Illustration of normal line.

*Consider the normal to the curve* $y = f(x)$ *at point* $A(x_1, y_1)$.

*The gradient of normal* $= -\dfrac{1}{f'(x)}$. *Thus, the equation of the normal is*

$$y - y_1 = -\frac{1}{f'(x)}(x - x_1).$$

***Example 4.33*** *Find the equation of the normal at* $A(1, 2)$ *on the curve* $f(x) = x^2 + 4x - 3$.

**Solution**

*Let* $f'(x) = 2x + 4$, *then the gradient of the normal at* $A(1, 2)$ *is given by* $-\dfrac{1}{2(1) + 4} = -\dfrac{1}{6}$. *Thus, the equation of the normal is*

$$y - 2 = -\frac{1}{6}(x - 1), \quad \textit{therefore} \quad x + 6y = 13.$$

## 4.9 Local extrema of functions

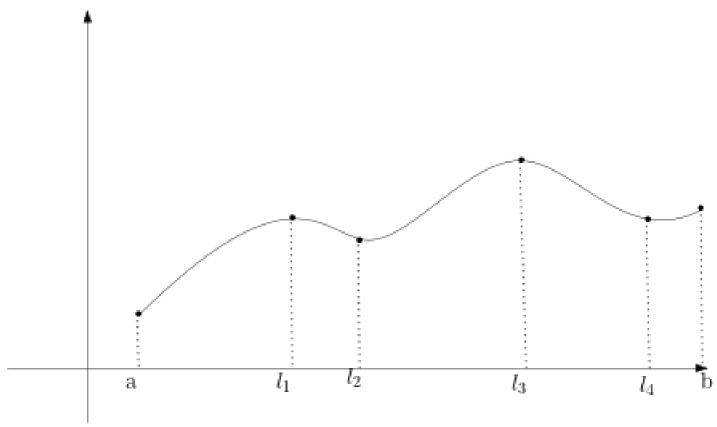



Figure 4.3: Illustration of local extrema of a function.

*Consider Figure 4.3. For the function whose graph is shown in Figure 4.3, the local maxima occurs at* $l_1$ *and* $l_3$, *whereas the local minima occurs at* $l_2$ *and* $l_4$.

***Definition 4.2*** *(local extrema) Let $c \in (a, b)$, then the funtion $f$ at the $c$ has the following extrema;*

1. *$f(c)$ is a local maximum of $f$ if $f(x) \leq f(c), \quad \forall x \in (a, b)$.*
2. *$f(c)$ is a local manimum of $f$ if $f(x) \geq f(c), \quad \forall x \in (a, b)$.*

***Definition 4.3*** *(Critical number) A number $c$ in the domain of a function $f$ is a critical number of $f$ if either $f'(c) = 0$ or $f'(c)$ does not exist.*

***Example 4.34*** *Find the critical numbers of $f(x) = x^{1/3}(8-x)$.*

**Solution**

*First, we determine the derivative of $f$,*

$$\begin{aligned} f'(x) &= \frac{1}{3}x^{-2/3}(8-x) - x^{1/3} \\ &= \frac{8-x}{3x^{2/3}} - x^{1/3} \\ &= \frac{8-x-3x}{3x^{2/3}} = \frac{8-4x}{3x^{2/3}}. \end{aligned}$$

*For the critical number, we set $f'(x) = 0$. This yields*

$$8 - 4x = 0 \quad \text{and} \quad x = 2.$$

*Also $f'(x)$ is undefined at $x = 0$. Hence the critical numbers are $0$ and $2$.*

### 4.9.1 Increasing and Decreasing functions

*Let $f$ be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval $(a, b)$.*

1. *if $f'(x) > 0$ on $(a, b)$, then $f$ is increasing on $[a, b]$.*
2. *if $f'(x) < 0$ on $(a, b)$, then $f$ is decreasing on $[a, b]$.*

### 4.9.2 The First Derivative test

*Suppose $c$ is a critical number of a function $f$ and $(a, b)$ is an open interval containing $c$. Suppose further that $f$ is differentiable on $(a, b)$, except possibly at $c$.*

1. *if $f'(x) > 0$ for $a < x < c$ and $f'(x) < 0$ for $c < x < b$, then $f(c)$ is a local maximum of $f$.*
2. *if $f'(x) < 0$ for $a < x < c$ and $f'(x) > 0$ for $c < x < b$, then $f(c)$ is a local minimum of $f$.*
3. *if $f'(x) > 0$ or $f'(x) < 0$ for al $x \in (a, b)$ except $x = c$, then $f(c)$ is not a local extremum of $f$.*

***Example 4.35*** *Find the local maxima and minima of $f$ if $f(x) = x^{1/3}(8 - x)$. State the intervals over which $f$ is increasing and decreasing.*

**Solution**

*For critical values, $f'(x) = 0$.*

$$\begin{aligned} f'(x) &= \frac{1}{3}x^{-2/3}(8 - x) + x^{1/3}(-1) \\ &= \frac{8 - x}{3x^{2/3}} - \frac{1}{3} \\ &= \frac{8 - x - 3x}{3x^{2/3}} = \frac{8 - 4x}{3x^{2/3}}. \end{aligned}$$

*For the critical number, we set $f'(x) = 0$. This yields*

$$8 - 4x = 0 \quad and \quad x = 2.$$

*Also $f'(x)$ is undefined at $x = 0$. Hence the critical numbers are 0 and 2.*

| *Interval* | $(-\infty,0)$ | $(0,2)$ | $(2,\infty)$ |
|---|---|---|---|
| *Sign of (8-4x)* | + | + | - |
| *Sign of $3x^{\frac{2}{3}}$* | + | + | + |
| *Sign of $f'(x)$* | + | + | - |
| *Direction* | $\nearrow$ | $\nearrow$ | $\searrow$ |

*Thus $f(x)$ has a local maximum at $x = 0$ and its value is*

$$f(2) = 2^{1/3}(8 - 2) = 2^{1/3}6 = 7.6,$$

*$f(x)$ is increasing on $(-\infty, 0) \bigcup (0, 2)$ and decreasing on $(2, \infty)$.*

### 4.9.3 Absolute minima and maxima(Extrema on a closed interval)

***Definition 4.4*** *Let a function $f$ be defined on an interval $I$ and let $c$ be a number in $I$.*

1. *$f(c)$ is the maximum value of $f$ on $I$ if $f(x) \leqslant f(c)$ for every $x$ in $I$.*
2. *$f(c)$ is the minimum value of $f$ on $I$ if $f(x) \geq f(c)$ for every $x$ in $I$.*

**Steps for finding absolute extrema**

*(1). Find all the critical numbers of $f$.*

*(2). Calculate $f(c)$ for each critical number $c$.*

*(3). Calculate $f(a)$ and $f(b)$.*

*(4). The absolute maximum and minimum of $f$ on $[a, b]$ are the largest and smallest of the functional values calculated in (2) and (3).*

***Example 4.36*** *If $f(x) = x^3 - 12x$, find the absolute maximum and minimum values of $f$ on the closed interval $[-3, 5]$.*

## Solution

*For critical values, $f'(x) = 0$.*

$$3x^2 - 12 = 0 \quad \textit{thus} \quad x^2 = 4. \quad \textit{Hence} \quad x = \pm 2.$$

*Thus, we solve for the functional values at the critical points and the end points of the interval.*

$$f(-2) = (-2)^3 - 12(-2) = -8 + 24 = 16,$$
$$f(2) = (2)^3 - 12(2) = 8 - 24 = -16,$$
$$f(-3) = (-3)^3 - 12(-3) = 9,$$
$$f(5) = 5^3 - 12(5) = 65.$$

*Thus the absolute maximum is* $f(5) = 65$ *and the absolute minimum is* $f(2) = -16$.

## 4.10 Concavity and the Second Derivative test

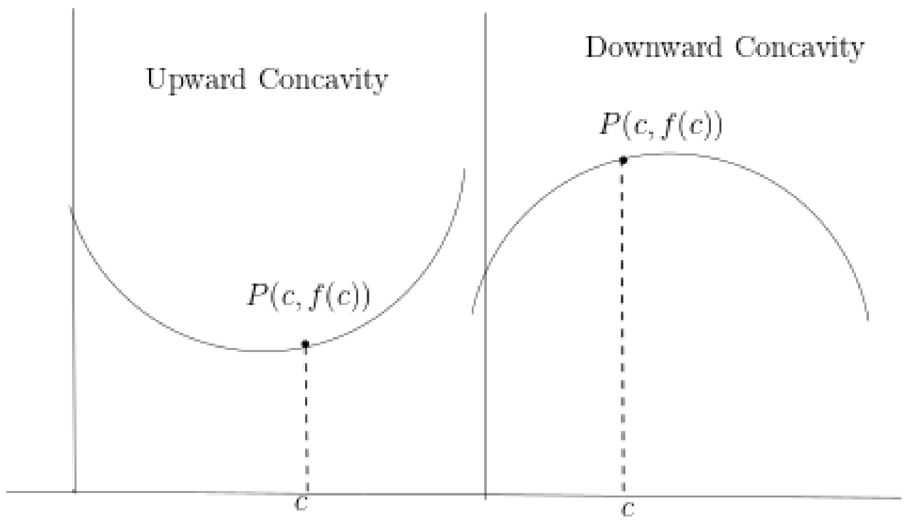



Figure 4.4: Illustration for Upward (left) and Downward (right) concavity.

### 4.10.1 Test for Concavity

*Suppose a function* $f$ *is differentiable on an open interval containing c, and* $f''(c)$ *exists.*

1. *if* $f''(c) > 0$ *on* $(a, b)$*, the graph is concave upward on* $(a, b)$.

2. *if $f''(c) < 0$ on $(a, b)$, the graph is concave downward on $(a, b)$.*

***Definition 4.5*** *(Point of Inflection) A point $P(k, f(k))$ on the graph of a function $f$ is a point of inflection if there exists an open interval $(a, b)$ containing $k$ such that one of the following statements holds.*

1. *$f''(x) > 0$ if $a < x < k$ and $f''(x) < 0$ if $k < x < b$; or*
2. *$f''(x) < 0$ if $a < x < k$ and $f''(x) > 0$ if $k < x < b$*

*Then the point $P(k, f(k))$ is an inflection point.*

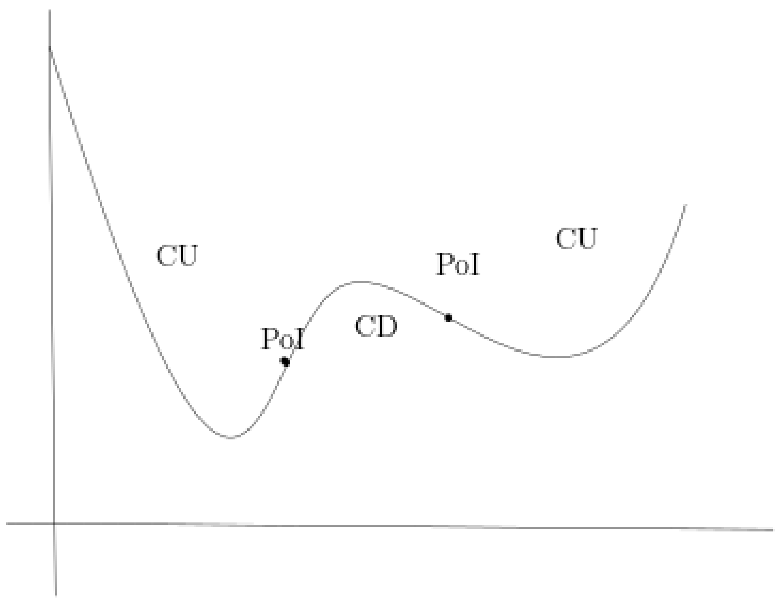



Figure 4.5: Illustration for Point of Inflection (POI).

### 4.10.2 The Second Derivative Test

*Suppose a function $f$ is differentiable on an open interval containing $c$ and $f'(c) = 0$*

1. *if $f''(c) < 0$, then $f$ has a local maximum at $c$.*
2. *if $f''(c) > 0$, then $f$ has a local minimum at $c$.*

***Example 4.37*** *If $f(x) = 12 + 2x^2 - x^4$, use the second derivative test to find the local maxima and minima of $f$. Discuss concavity, find the points of inflection.*

## Solution

*For the crtical values $f'(x) = 0$. The first derivative is given by*

$$f'(x) = 4x - 4x^3$$
$$4x - 4x^3 = 0 \quad \textit{therefore} \quad 4x(1 - x^2) = 0,$$
$$x = 0, 1 \quad \textit{and} \quad -1.$$

*Next, we determine the second derivative, i.e. $f''(x) = 4 - 12x^2$. Thus,*

$$f''(0) = 4 > 0 \quad \textit{and} \quad f''(1) = -8 < 0.$$

*Thus*

- *$f$ has a local maximum at $x = 1$ and also, $f''(-1) = -8 < 0$ $\implies$ $f$ has a local minimum at $x = 0$.*

*To locate the possible points of inflection, we solve the equation $f''(x) = 0$.*

$$4 - 12x^2 = 0.$$
$$\textit{Therefore} \quad x^2 = \frac{1}{3}. \quad \textit{Hence} \quad x = \pm\sqrt{\frac{1}{3}}.$$

*We have the following results.*

i. *Let $-1 \in \left(-\infty, -\sqrt{\frac{1}{3}}\right)$, $f''(-1) = -8 < 0$ concave downward.*

ii. *Let $0 \in \left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right)$, $f''(0) = 4 > 0$, concave upward.*

iii. *Let $1 \in \left(\sqrt{\frac{1}{3}}, \infty\right)$,$f''(1) = -8 < 0$ concave downward.*

*Finally, we determine the regions of concavity as follows.*

- $f$ *is concave downward on the interval* $\left(-\infty, -\sqrt{\frac{1}{3}}\right) \cup \left(\sqrt{\frac{1}{3}}, \infty\right)$ *and*
- $f$ *is concave upward on the interval* $\left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right)$.

*Hence* $f$ *has inflection points occurring at* $x = -\sqrt{\frac{1}{3}}$ *and* $x = \sqrt{\frac{1}{3}}$. *Thus, we have* $f\left(-\sqrt{\frac{1}{3}}\right) = \frac{113}{9}$ *and* $f\left(\sqrt{\frac{1}{3}}\right) = \frac{113}{9}$. *This means the inflection points are* $\left(-\sqrt{\frac{1}{3}}, \frac{113}{9}\right)$ *and* $\left(\sqrt{\frac{1}{3}}, \frac{113}{9}\right)$.

## 4.11 Optimization Problems

### 4.11.1 Steps In Solving Optimization Problems

i. *Understand the problem.*

ii. *Draw a diagram.*

iii. *Introduce notation. e.g.* $A = xy$.

iv. *Change the function to be optimized (minimized or maximized) to a function of one variable.*

v. *Find the appropriate extrema.*

***Example 4.38*** *A farmer has 2400ft of fencing and want to fence off a rectangular field that borders a straight river. What are the dimensions of the field that has the largest area?*

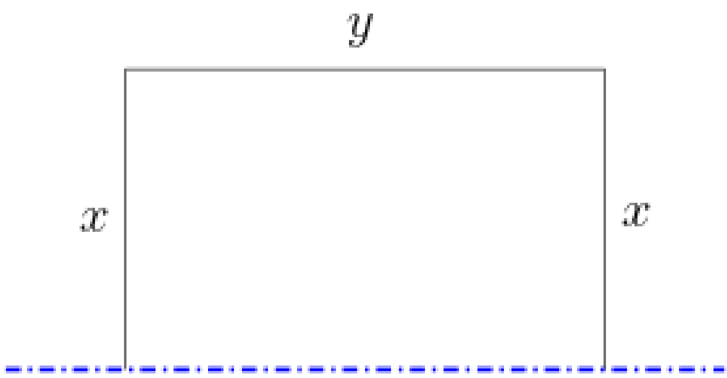



Figure 4.6: Illustration of the Example 4.38

**Solution**

*Let $x$ and $y$ denote the sides of the rectangle as shown above. Thus, Area of rectangular field $(A) = xy$. But*

$$2x + y = 2400 \quad thus \quad y = 2400 - 2x.$$
$$A = x(2400 - 2x) = 2400x - 2x^2.$$

*Note that $x \geq 0$ and $x \leq 1200$, otherwise $A < 0$. So the function that we wish to maximize is*

$$A = 2400x - 2x^2, \quad 0 \leq x \leq 1200,$$
$$A' = 2400 - 4x.$$

*For critical numbers, $A' = 0$*

$$2400 - 4x = 0$$
$$4x = 2400 \quad and \quad x = \frac{2400}{4} = 600.$$

*The maximum value of $A$ must occur either at this critical number or at an endpoint of the interval. Since,*

$$A(600) = 720,000ft^2 \quad A(0) = 0 \quad and \quad A(1,200) = 0.$$

*Thus, the maximum occurs at $x = 600ft$. This yields*

$$y = 2400 - 2(600) = 1200ft.$$

*Therefore, the dimension are $x = 600ft$ and $x = 1,200ft$.*

***Example 4.39*** *A right circular cylinder is inscribed in a sphere of radius R. If the height of the circular cylinder is* $2x$*, express its volume in terms of* $x$ *and* $R$*. Find the maximum value of this volume as* $x$ *varies.*

**Solution**

*Let* $O$ *denote the centre of the sphere and let* $r$ *denote the base radius of the cylinder, see Figure 4.7.*

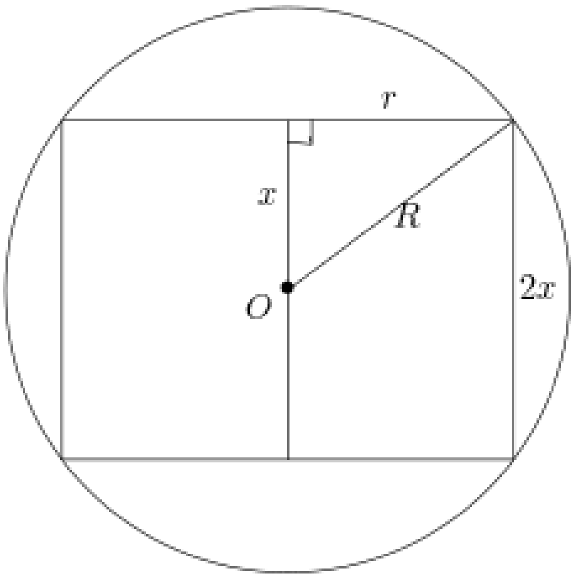



Figure 4.7: Illustration of the Example 4.39.

*The volume of the cylinder is given by*

$$V = \pi r^2(2x),$$

*but* $r^2 = R^2 - x^2$.

$$V = \pi(R^2 - x^2)(2x) = 2\pi(R^2x - x^3).$$

*For critical numbers,* $\dfrac{dV}{dx} = 0$*. Therefore,*

$$\frac{dV}{dx} = 2\pi(R^2 - 3x^2)$$

$$2\pi(R^2 - 3x^2) = 0 \quad and \quad x = \frac{R}{\sqrt{3}}.$$

*Next, the second derivative is given by*

$$\frac{d^2V}{dx^2} = 2\pi(-6x).$$

*Thus, if* $x = \frac{R}{\sqrt{3}}$. *Then,*

$$\frac{d^2V}{dx^2} = -12\pi\frac{R}{\sqrt{3}} < 0.$$

*Therefore,* $V$ *attains maximum at* $x = \frac{R}{\sqrt{3}}$. *The maximum is*

$$\begin{aligned} V &= 2\pi\left[R^2\left(\frac{R}{\sqrt{3}}\right) - \left(\frac{R}{\sqrt{3}}\right)^3\right] \\ &= 2\pi\left[\frac{3R^3 - R^3}{3\sqrt{3}}\right] \\ &= 2\pi\left[\frac{2R^3}{3\sqrt{3}}\right] = \frac{4\pi R^3}{3\sqrt{3}} = \frac{4\sqrt{3}\pi R^3}{9}. \end{aligned}$$

### 4.11.2 Related Rates

*The feature common to these applications is that the derivative dives the instantaneous rate of change of one quantity with respect to another. Suppose* $y$ *is described by a function of* $x$ *and* $t$ *is a time variable on which both* $x$ *and* $y$ *depend. Problems of related rates involve the relationship between* $\frac{dx}{dt}$ *and* $\frac{dy}{dt}$.

***Example 4.40*** *Air is being bumped into a spherical balloon so that its volume increases at a rate of* $100cm^3/s$. *How fasts is the radius of the balloon increasing when the diameter is* $50cm$*?*

**Solution**

*Let $V$ denote the volume of the balloon and $r$ its radius. Then $\frac{dV}{dt} = 100cm^3/s$ and $r = 25cm$. But the volume is given by $V = \frac{4}{3}\pi r^3$.*

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}.$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \cdot \frac{dV}{dt} = \frac{1}{4\pi (25)^2} \cdot 100 = \frac{1}{251}.$$

## 4.12 Exercises

1. *Find the derivative of the following functions;*

   *i.* $y = 10^{\left(\frac{1-x}{1+x}\right)}$

   *ii.* $y = e^{ax}\csc(bx+c)$

   *iii.* $y = \dfrac{e^{2x}\cos x}{x\sin x}$

   *iv.* $y = \log(3x^2+2)^5$

   *v.* $y = \ln(\sec x + \tan x)$

   *vi.* $y = \exp(\sqrt{\cot x})$

   *vii.* $y = \ln[\cos(\ln x)^x]$

   *viii.* $y = \cos x[\ln x]e^{x^2}x^x$

   *ix.* $y = \ln\left(\dfrac{a + b\tan x}{a - b\tan x}\right)$

   *x.* $y = (\tan x)^{\cot x} + (\sec x)^x$

   *xi.* $y = \ln(x^{\ln x} + \sin^{\ln x} x)$

   *xii.* $y = \dfrac{(4x^2-1)(1+x^2)^{\frac{1}{2}}}{x^3(x-7)^{\frac{3}{4}}}.$

2. *If $y = x^{x^{x^{\cdots}}}$, prove that $x\dfrac{dy}{dx} = \dfrac{y^2}{1 - y\ln x}$.*

3. *Show that $\dfrac{dy}{dx} = \dfrac{y^2\cot x}{1 - y\ln(\sin x)}$,*
   *if $y = (\sin x)^{(\sin x)^{(\sin x)^{\cdots}}}$.*

4. *If $y = \sqrt{\ln x + \sqrt{\ln x + \sqrt{\ln x + \cdots}}}$,*
   *show that $\dfrac{dy}{dx} = \dfrac{1}{x(2y-1)}$.*

5. *If* $x^y = \exp(x - y)$, *prove that* $\dfrac{dy}{dx} = \dfrac{\ln x}{(1 + \ln x)^2}$.

6. *Find* $\dfrac{dy}{dx}$ *at* $(0, 1)$
   *if* $\dfrac{(\ln x)^2}{e^y} - \ln x + y = e^{\sin(2x-2)\cos^2 y} - 1$

7. *Find the second derivative of the following functions;*
   (a) $y = \cos(\ln x)$
   (b) $y = x^x$
   (c) $y = a^x x^a$

8. *If* $x^3 - y^3 = 1$, *show that* $y^5 y'' + 2x = 0$

9. *Find the equation of the tangent and normal line to the graph of* $x^3 - x\ln y + y^3 = 2x + 5$ *at the point where* $x = 2$.

10. *Find the equation of tangent and normal to* $f(x) = \frac{x}{e}\ln(\frac{e}{x})$ *when* $x = e$.

11. *Find the smallest possible value of the consatnt k such that the graphs of* $y = e^x$ *and* $y = ksinx$ *are tangent to one another. Find also the point of tangency.*

12. *For what nonnegative value(s) of b is the line* $y = \frac{-1}{12}x + b$ *normal to* $y = x^3 + \frac{1}{3}$?

13. *Find the value of k, where k is a constant, such that the graph of* $y = k^x$ *and* $y = \log kx$ *will be tangent to one another. Find the point of tangency.*

14. *A certain point(s)* $(a, b)$ *is on the graph of* $y = x^3 + x^2 - 9x - 9$, *and the tangent line to the graph at* $(a, b)$ *passes through the point* $(4, -1)$. *Find* $(a, b)$.

15. *At what point(s) of the graph of* $y = x^5 + 4x - 3$ *does the tangent line to the graph also pass through the point* $B(0, 1)$?

16. *If the line* $4x - 9y = 0$ *is tangent in the first quadrant to the graph of* $y = \frac{1}{3}x + c$, *what is the value of c?*

17. *If* $y = xe^x - e^x - 2x^2$*, determine any relative maximum or minimum values. Is there an absolute minimum or maximum value?*

18. *Suppose* $h = e^{\ln t} - \ln(t^2 - 1)$*. ( t in seconds and h in metres)*

    i. *When is the velocity equal to zero?*

    ii. *Is this a maximum or minimum height?*

    iii. *Find the acceleration.*

    iv. *Is there a minimum velocity?*

19. *Given a function* $f(x) = x^{\frac{3}{5}}(4 - x)$

    (a) *Find the critical numbers.*

    (b) *Find interval of decrease and/or increase of f*

    (c) *Find the local extrema if there are.*

20. *find the critical numbers of* $f(x) = (x - 1)^{\frac{2}{3}}$ *and determine whether they yield relative extrema or inflection points.*

21. *Find the absolute extrema of the following functions on their respective interval.*

    (a) $f(x) = \sin x + x$ *on*$[0, 2\pi]$.

    (b) $f(x) = \dfrac{2x + 5}{(x^2 - 4)^2}$ *on* $[-5, -3]$.

    (c) $f(x) = \cos^2 x + \sin x$ *on* $[0, \pi]$

    (d)
    $$f(x) = \begin{cases} x^3 - \frac{x}{3} & for \quad 0 \leq x \leq 1 \\ x^2 + x - \frac{4}{3} & for \quad 1 < x \leq 2 \end{cases}$$

22. *A rectangular box with an open top is to be constructed from a rectangular sheet of cardboard measuring* $20cm$ *by* $12cm$ *by cutting equal squares of side length* $x$ *cm out of the four corners and folding the flaps up.*

    (a) *Express the volume as a function of* $x$.

(b) *Determine the dimensions of the box with greatest volume and give this maximum volume to the nearest whole number.*

23. *On a warm day in a garden, water in a bird bath evaporates in such a way that the volume, $VmL$, at time $t$ hours is given by* $V = \dfrac{60t+2}{3t}, t > 0$.

    (a) *Show that* $\dfrac{dV}{dt} < 0$

    (b) *At what rate is the water evaporating after* 2 *hours?*

24. *The new owner of an apartment want to install a window in the shape of a rectangle surmounted by a semicircle in order to allow more light into the apartment. The owner has* 336*cm of wood for a surround to the window and wants to determine the dimensions that will allow as much light into the apartment as possible. Given the radius of the semicircle to be $x$ cm and height to be $h$ cm;*

    (a) *Show that the area $A$, in $cm^2$, of the window is* $A = 336x - \frac{1}{2}(4+\pi)x^2$.

    (b) *Hence determine, to the nearest cm, the width and the height of the window for which the area is greatest.*

    (c) *Structural problems require that the width of the window should not exceed* 84*cm. What should the new dimensions of the window be for maximum area?*

25. *The total surface area of a closed cylinder is* $200cm^2$. *If the base radius is $r$ cm and the height is $h$ cm:*

    (a) *Express $h$ in terms of $r$.*

    (b) *Show that the volume, $Vcm^3$, is* $V = 100r - \pi r^3$

    (c) *Hence show that for a minimum volume, the height must equal the diameter of the base.*

    (d) *Calculate, correct to the nearest integer, the minimum volume if* $2 \leq r \leq 4$.

26. *A right circular cone is inscribed in a sphere of radius* $7cm$. *If* $r$ *and* $h$ *are the radius and height respectively,*

    (a) *Show that the volume* $V cm^3$ *of the cone satisfies the relationship* $V = \frac{1}{3}(14h^2 - h^3)\pi$.

    (b) *Hence, obtain the exact values of* $r$ *and* $h$ *for which the volume is greatest, justifying your answer.*

27. *A container in the shape of an inverted right cone of radius* $2cm$ *and depth* $5cm$ *is being filled with water. When the depth of water is* $hcm$, *the radius of the water level is* $rcm$.

    (a) *Express the volume of the water as a function of* $h$

    (b) *At what rate, with respect to the depth of water, is the volume of water changing, when its depth is* $1cm$*?*

28. *A cylinderical tank of radius* $10feet$ *is being filled with wheat at the rate of* $314\,cubic\,feet\,per\,minute$. *How fast is the depth of the wheat increasing?*

29. *A* $5 - foot$ *girl is walking toward a* $20 - foot$ *lamppost* $t$ *the rate of* $6\,feet\,per\,second$. *How fast is the tip of her shadow(cast by the lamp) moving?*

30. *A rocket is shot vertically upward with an initial velocity of* $400cm\,per\,second$. *Its height* $s$ *after* $t$ *seconds is* $s = 400t - 16t^2$. *How fast is the distance changing from the rocket to an observer on the ground* $1800cm$ *away from the laundering site, when the rocket is still rising and is* $2400cm$ *above the ground?*

# Chapter 5

# ANTIDERIVATIVES

***Definition 5.1*** *A function $F$ is an antiderivative of the function $f$ if $F'(x) = f(x)$. For example, the function $F(x) = x^2$ is an antiderivative of $f(x) = 2x$, because*

$$F'(x) = 2x = f(x).$$

*There are many other antiderivatives of $2x$, such as $x^2+2$, $x^2-\frac{5}{3}$ and $x^2+\sqrt{3}$. In general, if $c$ is any constant, then $x^2+c$ is an antiderivative of $2x$ because*

$$\frac{d}{dx}(x^2+c) = 2x + 0 = 2x.$$

*Thus there is a family of antiderivatives of $2x$ of the form $F(x) = x^2+c$, where c is any constant. The process of finding antiderivatives is called* ***`antidifferentiation`*.**

## 5.1 Indefinite Integrals

*The notation*

$$\int f(x)dx = F(x) + c,$$

*where $F'(x) = f(x)$ and $c$ is an arbitrary constant, denotes the family of all antiderivatives of $f(x)$. The symbol "$\int$" is an integral sign. We call $\int f(x)dx$ the indeifinite integral of $f(x)$.*

*The expression $f(x)$ is called the integrand, and $c$ is the constant of integration. The process of finding $F(x)+c$, when given $\int f(x)dx$ is referred to as* ***indefinite integration****.*

***Example 5.1*** *Evaluate (i) $\int 3x^2dx$ and (ii). $\int \cos x dx$.*

**Solution**

*(i).* $\int 3x^2dx = x^3 + c.$

*(ii).* $\int \cos x dx = \sin x + c.$

### 5.1.1 Some Basic Integration Functions

1. $\int k dx = kx + c$
2. $\int x^n dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$
3. $\int \cos x dx = \sin x + c$
4. $\int \sin x dx = -\cos x + c$
5. $\int \sec^2 x dx = \tan x + c$
6. $\int \csc^2 x dx = -\cot x + c$
7. $\int \sec x \tan x dx = \sec x + c$
8. $\int \csc x \cot x dx = -\csc x + c$
9. $\int e^x dx = e^x + c$
10. $\int \frac{1}{x} dx = \ln x + c$

***Example 5.2*** *Evaluate*

a. $\int x^8 dx$

b. $\int \frac{1}{x^3} dx$

c. $\int \sqrt[3]{x^2} dx$

d. $\int \frac{\tan x}{\sec x} dx$

**Solution**

1. $\int x^8 dx = \frac{x^9}{9} + c$

2. $\int \frac{1}{x^3}dx = \int x^{-3}dx = \frac{x^{-2}}{-2} + c = -\frac{1}{2x^2} + c$

3. $\int \sqrt[3]{x^2}dx = \int x^{2/3}dx = \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + c = \frac{3x^{5/3}}{5} + c$

4. $\int \frac{\tan x}{\sec x}dx = \int \cos x\left(\frac{\sin x}{\cos x}\right)dx = \int \sin x dx = -\cos x + c$

***Remark 5.1***

$$\int \frac{d}{dx}(f(x))dx = f(x) + c.$$

***Example 5.3*** *Evaluate*

$$\int \frac{d}{dx}(x^2)dx$$

**Solution**

$$\int \frac{d}{dx}(x^2)dx = x^2 + c.$$

### 5.1.2 Properties of Indefinite Integral

1. $\int cf(x)dx = c\int f(x)dx$, *for any non-zero constant c.*

2. $\int [f(x) \pm g(x)]dx = \int f(c)dx \pm \int g(x)dx$

***Example 5.4*** *Evaluate (a).* $\int (5x^3 + 2\cos x)dx$ *(b).* $\int (8x^3 - 6\sqrt{x} + \frac{1}{x^3})dx$

**Solution**

*a.*

$$\int (5x^3 + 2\cos x)dx = \int 5x^3 dx + \int 2\cos x dx$$

$$
\begin{aligned}
&= 5\int x^3 dx + 2\int \cos x dx \\
&= 5\left(\frac{x^4}{4}\right) + 2\sin x + c \\
&= \frac{5x^4}{4} + 2\sin x + c.
\end{aligned}
$$

*b.*

$$
\begin{aligned}
\int (8x^3 - 6\sqrt{x} + \frac{1}{x^3})dx &= \int 8x^3 dx - \int 6\sqrt{x}dx + \int \frac{1}{x^3}dx \\
&= \frac{8x^4}{4} - \frac{6x^{3/2}}{\frac{3}{2}} - \frac{1}{2x^2} + c \\
&= 2x^4 - 4x^{3/2} - \frac{1}{2x^2} + c.
\end{aligned}
$$

***Example 5.5*** *Evaluate (a).* $\int \frac{(x^2-1)^2}{x^2}dx$ *(b).* $\int \frac{1}{\cos x \cot x}dx$

**Solution**

*a.*

$$
\begin{aligned}
\int \frac{(x^2-1)^2}{x^2}dx &= \int \frac{x^4 - 2x^2 + 1}{x^2}dx \\
&= \int x^2 - 2 + \frac{1}{x^2}dx = \frac{x^3}{3} - 2x - \frac{1}{x} + c.
\end{aligned}
$$

*b.*

$$
\begin{aligned}
\int \frac{1}{\cos x \cot x}dx &= \int \frac{\sin x}{\cos^2 x}dx \\
&= \int \sec x \tan x dx = \sec x + c.
\end{aligned}
$$

## 5.2 The Definite Integral

*The following is the basic tool for evaluating definite integrals.*

### 5.2.1 Fundamental Theorem of Calculus

*If $f$ is continuous at every point $[a, b]$ and if $F$ is an antiderivative of $f$ on $[a, b]$, then*

$$\int_a^b f(x)dx = F(b) - F(a).$$

*Some other common notations are*

$$\int_a^b f(x)dx = F(x)\Big|_a^b \quad \textit{and} \quad \int_a^b f(x)dx = F(x)\Big|_{x=a}^b.$$

*'a' is known as the lower limit of the integral and 'b' is the upper limit of integration.*

***Example 5.6*** *Evaluate*

$$\int_1^2 x\,dx = \frac{x^2}{2}\Big|_1^2 = \frac{(2)^2}{2} - \frac{(1)^2}{2} = 2 - \frac{1}{2} = \frac{3}{2}$$

## 5.3 Change of Variables in Indefinite Integrals

*The basic integration formulas cannot be used directly to evaluate integrals such as*

$$\int \sqrt{5x+7}dx \quad \textit{or} \quad \int \cos 4x\,dx.$$

*In this section, we shall develop a simple but powerful method for changing the variable of integration so that these integrals can be evaluated by the basic formulas.*

***Example 5.7*** *To illustrate the procedure, consider the indefinite integral,*

$$\int \sqrt{5x+7}\,dx.$$

*Let* $u = 5x + 7$ *and calculate* $du = 5dx$. *This yields* $dx = \frac{1}{5}du$.

$$\begin{aligned}\int \sqrt{5x+7}dx &= \int \frac{1}{5}\sqrt{u}\,du \\ &= \frac{1}{5}\int u^{1/2}du = \frac{1}{5}u^{3/2\,(2/3)} + c \\ &= \frac{2}{15}(5x+7)^{3/2} + c.\end{aligned}$$

***Example 5.8*** *Evaluate* $\int \frac{x^2-1}{(x^3-3x+1)^6}\,dx$

**Solution**

*Let* $u = x^3 - 3x + 1$, *then*

$$du = (3x^2 - 3)\,dx = 3(x^2-1)\,dx.$$

*This implies* $\frac{du}{3} = (x^2-1)dx$. *Thus by substitution, we have*

$$\begin{aligned}\int \frac{x^2-1}{(x^3-3x+1)^6}\,dx &= \frac{1}{3}\int u^{-6}\,du \\ &= -\frac{1}{3}\cdot\frac{u^{-5}}{5} + c = -\frac{1}{15u^5} + c \\ &= -\frac{1}{(x^3-3x+1)^5} + c.\end{aligned}$$

***Example 5.9*** *Evaluate* $\int \cos^3 5x\,\sin 5x\,dx$.

**Solution**

*Let* $u = \cos 5x$, *then*

$$du = (-5\sin 5x)dx \quad \textit{and} \quad -\frac{1}{5}du = \sin 5xdx.$$

*Thus, by substitution, we have*

$$\begin{aligned}\int \cos^3 5x \sin 5x \, dx &= -\frac{1}{5}\int u^3 \, du \\ &= -\frac{1}{5} \cdot \frac{u^4}{4} + c = -\frac{\cos^4 5x}{20} + c.\end{aligned}$$

***Example 5.10*** *Evaluate (a).* $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx$ *and (b).* $\int \frac{\ln x}{x} \, dx.$

**Solution**

*(a). Let* $u = \sqrt{x}$*. Then*

$$du = \frac{1}{2\sqrt{x}} \, dx \quad \text{and} \quad 2du = \frac{1}{\sqrt{x}} \, dx.$$

$$\begin{aligned}\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx &= 2\int e^u \, du \\ &= 2e^u + c = 2e^{\sqrt{x}} + c.\end{aligned}$$

*(b). Let* $u = \ln x$*. Then*

$$du = \frac{1}{x} dx.$$

*Therefore, by substitution, we have*

$$\begin{aligned}\int \frac{\ln x}{x} \, dx &= \int u \, du \\ &= \frac{u^2}{2} + c = \frac{(\ln u)^2}{2} + c = \frac{(\ln (\ln x))^2}{2} + c.\end{aligned}$$

***Example 5.11*** *Evaluate* $\int \cos 4x \, dx.$

## Solution

*Let* $u = 4x$. *Then*

$$du = 4dx \quad \textit{and} \quad dx = \frac{1}{4}du$$

*Thus, by substitution, we have*

$$\begin{aligned}\int \cos 4x\, dx &= \frac{1}{4}\int \cos u\, du \\ &= \frac{1}{4}\sin u + c = \frac{1}{4}\sin 4x + c.\end{aligned}$$

***Example 5.12*** *Evaluate* $\int (2x^3 + 1)^7 x^2\, dx$

## Solution

*Let* $u = 2x^3 + 1$. *Then*

$$du = 6x^2 dx \quad \textit{and} \quad \frac{1}{6}du = x^2 dx.$$

*Thus, by substitution, we have*

$$\begin{aligned}\int (2x^3 + 1)^7 x^2\, dx &= \frac{1}{6}\int u^7\, du \\ &= \frac{1}{6}\cdot\frac{u^8}{8} + c \\ &= \frac{(2x^3+1)^8}{48} + c\end{aligned}$$

*Let us consider an example of definite integral with changing variable.*

***Example 5.13*** *Evaluate* $\int_2^{10} \frac{3}{\sqrt{5x-1}}\, dx$.

## Solution

*Let* $u = 5x - 1$, *then* $du = 5dx$. *This implies* $dx = \frac{1}{5}du$. *If* $x = 2$, *then* $u = 10 - 1 = 9$. *If* $x = 10$, *then* $u = 50 - 1 = 49$.

$$\begin{aligned}
\int_2^{10} \frac{3\,dx}{\sqrt{5x-1}} &= \int_9^{49} \frac{3}{u^{1/2}}\left(\frac{1}{5}\,du\right) \\
&= \frac{3}{5}\int_9^{49} u^{-1/2}\,du \\
&= \frac{3}{5} \times \frac{2u^{1/2}}{1}\Bigg|_{u=9}^{49} \\
&= \frac{6}{5}(49)^{1/2} - \frac{6}{5}(6)^{1/2} \\
&= \frac{6(7)}{5} - \frac{6(3)}{5} = \frac{42-18}{5} = \frac{24}{5}
\end{aligned}$$